APPLICATIONS OF THE CALCULUS TO MECHANICS

HEDRICK AND KELLOGG

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PREFACE

Wherever the teaching of mathematics to engineering students is discussed, and frequently in cases of other classes of students, the criticism which is almost without exception the most insistent is this: that the student leaves the course without adequate ability to *apply* his mathematical knowledge.* This means that he has not the faculty of taking a problem, *giving it an analytic formulation,* and *interpreting the analytic results.* It is an open question whether it is the duty of the teacher of mathematics, or of the teacher of the more technical work which involves mathematics, to supply the needed training, but usually the mathematician is glad at least to share the responsibility and to do whatever he can to make his work fruitful, fully conscious of the fact that if he can successfully make the contact of his subject with the problems of the laboratory, of the engineering office, and of other activities, he will thereby add immensely to the vitality and interest of his work. With such a motive, it has been the practice at the University of Missouri to follow the course in sophomore calculus with several weeks in applications to mechanics, this being a subject rich in the kind of material desired. The present book is a formulation of the work there attempted, and it is believed that the need at our institution which has called the book into being will make its appearance welcome to a large number of mathematical departments.

Opinions will differ as to the subject-matter which such a book should contain. The authors were guided by the feeling that it was practice in applying calculus rather than a broad knowledge

* See, for instance, the reports of the joint meeting of mathematicians and engineers held in Chicago, December, 1907, under the auspices of the Chicago Section of the American Mathematical Society. These reports appeared in *Science* during the ensuing year.

of mechanics that was desired, and that such an end would be hindered rather than helped by a wide diversity of subject-matter. It was felt that, when feasible, the student secures a better insight into a subject by developing a portion of the theory himself, and so "exercises" have been introduced which form a part of such development of the theory. The "problems" are the applications of the theory, usually to cases in which numerical data are given. They vary considerably in difficulty, but it is thought that they are sufficiently numerous to supply a good number of the proper grade for any given class; and, furthermore, it is believed that a more difficult problem with a proper amount of elucidation in advance by the teacher will be of far more value to the student than a number of problems so well within his range of ability as to require very little study on his part concerning the method of attack. Some hints will be found in the text, and some suggestions are given among the answers at the end of the book.

We have endeavored, by a judicious selection of the problems to which answers are given, to secure a safe mean between the evil of supplying detailed answers, which rob the student of his independence, and the evil of furnishing him with no check upon his work. We shall be most grateful for any corrections or suggestions concerning this or other aspects of the book.

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APPLICATIONS OF THE CALCULUS TO MECHANICS

CHAPTER I

INTRODUCTION

1. Mechanics. Mechanics deals with the position or motion of bodies in space; so that the ideas of *space* and *time* implied in motion, and of *mass* implied in body, are fundamental. These three concepts cannot be defined, for there is nothing simpler in terms of which we can define them.

2. Units. With each of the three concepts mentioned is associated the idea of quantity, the quantity being measured by comparison with a conventional unit or standard quantity. In Great Britain, and for commercial purposes in the United States, the units of space, mass, and time generally employed are the foot, pound, and second respectively. The initials of these fundamental units are usually used to designate the system, "the F.P.S. system." In France, and for scientific purposes in the United States, the units are the centimeter, gram, and second respectively, giving "the C.G.S. system." The following table gives the important derived units of velocity, acceleration, and force. In engineering circles other units of mass are in use, with the result that the units of force are also changed. The "engineering" or "technical" or "gravitational" units of mass and force are also given below.*

* For further information concerning units, see Ziwet, *Theoretical Mechanics;* Encyclopedia Britannica on "Weights and Measures." In the following we shall meet with other derived units, but these will be defined as they arise.

Quantity	British or F.P.S. System	French or C.G.S. System
Velocity = time-rate of change of space . .	One foot per second.	One centimeter per second.
Acceleration = time-rate of change of velocity . . .	One foot per second per second. (The acceleration of a falling body near the earth's surface is $g = 32.2$ of these units, nearly.)	One centimeter per second per second. (The acceleration of a falling body near the earth's surface is $g = 981$ of these units, nearly.)
Force = Mass × Acceleration	One poundal: the force which, acting constantly throughout a second, will give to a pound of matter a unit acceleration, that is, increase the velocity by one foot per second.	One dyne: the force which, acting constantly throughout a second, will give to a gram of matter a unit acceleration, that is, increase the velocity by one centimeter per second.

The engineering units of mass and force in the two systems follow:

Quantity	British or F.P.S. System	French or C.G.S. System
Mass	The mass of a body weighing g pounds. (To determine the mass of a body in technical units, divide its weight in pounds by $g = 32.2$.)	The mass of a body weighing g kilograms. (To determine the mass of a body in technical units, divide its weight in kilograms by $g = 981$.)
Force = Mass × Acceleration	One pound: the force which the earth exerts at its surface upon a body weighing one pound. (One pound = g poundals.)	One kilogram: the force which the earth exerts at its surface upon a body weighing one kilogram. (One kilogram = 1000 g dynes.)

One foot = 30.5 centimeters.	One centimeter = .0328 feet.
One pound = .373 kilograms.	One kilogram = 2.68 pounds.

I. PROBLEMS ON UNITS

1. A force equal to the weight of 30 gm. acts upon a mass of 2 kgm. What is the acceleration in C.G.S. and in F.P.S. units?

Solution. The equation $f = ma$ holds if f is measured in dynes and m in grams. It may be solved for a, thus giving the desired result. The weight of 30 gm. is a force of 30 g dynes, and 2 kgm. equals 2000 gm. Hence $30\,g = 2000\,a$, and $a = 30\,g/2000 = 30 \times 981/2000$, which is approximately 14.7 cm. per sec. per sec. As one centimeter is .0328 in., this is $14.7 \times .0328$, or .482 ft. per sec. per sec.

2. A man walks 5 mi. per hour. What is his speed in F.P.S. units, and what in C.G.S. units?

3. A mass of 25 gm. moves with an acceleration of 30 cm. per sec. per sec. What is the force acting, measured in C.G.S. and in F.P.S. units?

4. If sound travels 1000 ft. per sec. in air, what is its speed measured in C.G.S. units?

5. A boy throws a ball with a velocity of 50 m. per sec. What is its speed in F.P.S. units?

6. A force of 6 lb. acts upon a 3-lb. weight. What is the acceleration in F.P.S. and C.G.S. units?

3. Vectors. The units above apply primarily to bodies moving in a straight line. In the case of bodies not moving in a straight line, velocity, acceleration, and force cannot be characterized by one measurement alone, say of their magnitude; but in addition their direction must be specified. They are examples of what are called *vectors,* and are usually represented geometrically by directed straight line segments.

4. The fixing of vectors. For the fixing of vectors in space three magnitudes are necessary. These may be selected in various ways; we mention here only two of them. Let us denote the vector by V. It is to be noted then that V does not stand for a number in the ordinary sense, but for a set of numbers. The first way of fixing V is by its magnitude, v, essentially a positive number, and the angles α, β, and γ which it makes with the coördinate axes (see Fig. 1). These three angles are not independent, for $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$; so that V involves but three independent magnitudes, v and two of the three angles α, β, and γ.

The relation given follows from the law for the diagonal of a cuboid: $v^2 = v_x^2 + v_y^2 + v_z^2$, whence, dividing by v^2, we have

$$1 = (v_x/v)^2 + (v_y/v)^2 + (v_z/v)^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma.$$

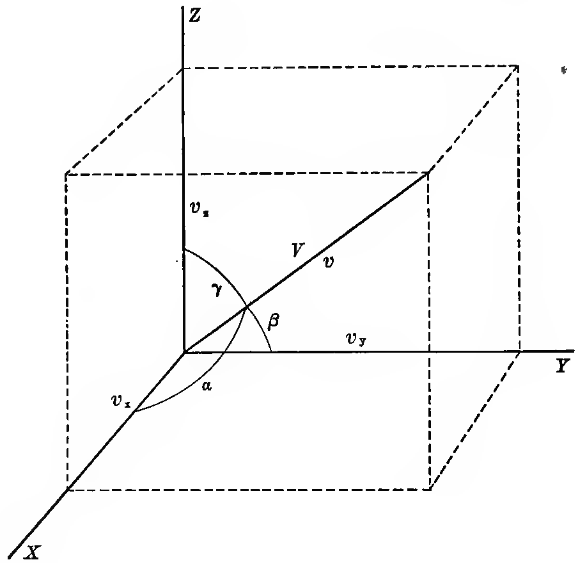


FIG. 1

Another means of fixing V, and that usually used in analytic treatments of vectors, is by its projections on the axes, v_x, v_y, v_z.* If the initial point of the vector be placed at the origin, these projections are simply the coördinates of the terminal points.

Ex. 1. Express v_x, v_y, v_z in terms of v, α, β, and γ, and conversely.

Ex. 2. If our vector is required to lie in a fixed plane, two quantities suffice to fix it. Give two pairs of quantities analogous to the two sets given above for space, and give the relations between them.

5. Operations upon vectors. If an object move from a point A on a table to a point B on the table, while at the same time the table moves so that the point B passes to a point C fixed in the room, the body originally at A will arrive at C, and this no matter

* If l stands for a given direction, v_l stands for the projection of the vector V upon a line with that direction.

how the separate motions are carried out. Now the motion of the body relative to the table may be represented by a vector V_1, while the motion of the table may be represented by a vector V_2, and the total effect also amounts to a vector V_3. The last is said to be the *sum* of the first two, and this illustrates the general definition: *place the initial point of one vector upon the terminal point of the other; the vector running from the free initial point to the free terminal point is called the sum, or resultant vector.*

Ex. 1. Show that V_3 will be the same whether the initial point of V_2 be put upon the terminal point of V_1 or the initial point of V_1 be put upon the terminal point of V_2.

Ex. 2. Show that another method of adding vectors, equivalent to the above, is to put the initial points of V_1 and V_2 together, to complete the parallelogram, and to draw the diagonal from the common initial point. This diagonal will be V_3.

Ex. 3. How would you obtain the sum of 3 vectors? of any given number of vectors? Show that n vectors may be added in any order.

On the other hand, a given vector may be *resolved* into two vectors whose sum or resultant it is. This may be done in an unlimited number of ways. Several especially important general ways are indicated in the following exercises.

Ex. 4. Let there be given in a plane a vector V and two nonparallel lines l_1 and l_2. Show that there are two vectors, V_1 parallel to l_1 and V_2 parallel to l_2, whose sum is V. Give the construction for V_1 and V_2. Is there more than one solution?

Ex. 5. Let there be given in space a vector V and three lines l_1, l_2, and l_3, not in the same plane or parallel. Show that there are three vectors, one parallel to l_1, one to l_2, and one to l_3, whose sum is V. Is there more than one solution? To get a good figure, draw parallels to l_1, l_2, and l_3 through the origin of V, so that one looks into the solid angle formed.

Ex. 6. Let there be given a vector V and a vector V_1. Show how to construct a vector V_2 such that $V_1 + V_2 = V$.

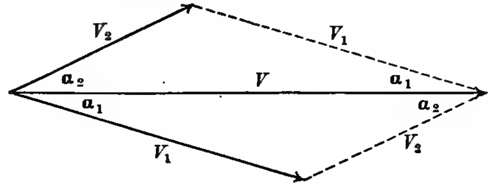


FIG. 2

Ex. 7. Let V_1 and V_2 be two vectors whose resultant is V, and let their magnitudes be v_1, v_2, and v respectively; let a_1 and a_2 be the positive angles between V_1 and V and between V_2 and V respectively;

(a) Show that the magnitude v of the resultant is the square root of the sum of the squares of the magnitudes of the components increased by twice their product times the cosine of the angle $(a_1 + a_2)$ between them, that is, $v^2 = v_1^2 + v_2^2 + 2\,v_1v_2 \cos(a_1 + a_2)$.

(b) Show that $\sin a_1/v_2 = \sin a_2/v_1$.

(c) Show that $\tan a_1 = [v_2 \sin(a_1 + a_2)]/[v_1 + v_2 \cos(a_1 + a_2)]$.

Hint. The first two parts are essentially formulas from plane trigonometry. The third should be obtained from the figure by dropping a perpendicular from the tip of V to V_1 produced. These results are useful in finding direction and magnitude of resultants.

Ex. 8. Take coördinate axes so that V lies on the x-axis beginning with the origin. Find the projections on the axes of all the vectors, and thus prove analytically by the methods of Analytic Geometry the formulas of Ex. 7, (a) and (b).

Hint. Show that $v = v_1 \cos a_1 + v_2 \cos a_2$, $0 = v_1 \sin a_1 - v_2 \sin a_2$; then square and add.

It should be noted that in the above statements concerning composition and resolution of vectors it must be possible to regard the vectors as acting at the same point, or as *concurrent.* The following important facts concerning operations with vectors should be clearly understood by the student.

I. *The projections upon each coördinate axis of the sum of two vectors is the sum of the projections upon that axis of the two component vectors. The same is true for the projections upon any line.*

Ex. 9. Prove the above theorem, making use of the "projection" theorem of the Trigonometry or Analytic Geometry text-books. Draw your own figure.

Ex. 10. Extend the above theorem to n vectors (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), $\cdots$, (X_n, Y_n, Z_n), with resultant (X, Y, Z). That is, show that $X = X_1 + X_2 + X_3 + \cdots + X_n$, etc.

Such a sum is usually abbreviated in mathematical writing by ΣX_i, the Greek sigma, Σ, denoting a sum.

II. *The projections on each axis of k times a vector, where k is a number, is k times the projection upon the axis in question of the vector. The same is true of the projection upon any line.*

Ex. 11. This amounts to a definition of *product of a vector by a number.* Verify the fact that it holds when k is a positive integer and the multiplication is regarded as repeated addition. Show that in all cases the product vector has the same direction as the given one, and that its magnitude is k times that of the given one.

If a vector vary with the time t, or any other parameter, its derivative may be defined, for we know how to subtract vectors and to divide by numbers. The notion of *limit* of a set of vectors will be sufficiently clear. The derivative will be found to have the property:

III. *The projection upon each axis of the derivative of a vector is the derivative of the projection upon the axes in question of the vector. The same is true of the projection upon any line.*

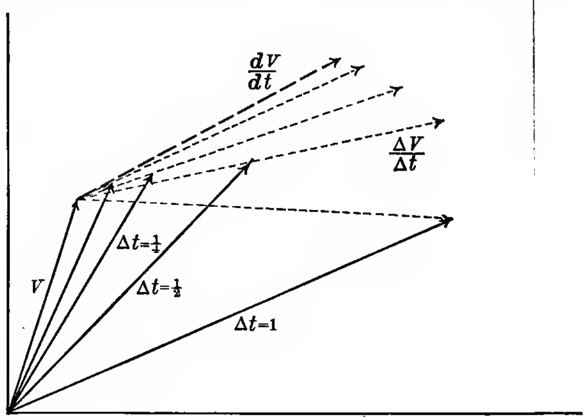


FIG. 3

Ex. 12. Working in two dimensions, draw the variable vector V for various values of t, approaching t_0, and construct the appropriate difference vectors, multiplying their length by $1/\Delta t$. The set of vectors thus obtained should have a limiting position, which we call the derivative vector. Find the magnitudes of the various approximating vectors and show that the limit of these magnitudes is $\sqrt{(dv_x/dt)^2 + (dv_y/dt)^2}$. Next show that the limit of their slope is $(dv_y/dt) \div (dv_x/dt)$. Then noting that these are the magnitude and slopes of the vector whose projections are the derivatives of the projections of V, the truth of the above statement, III, is apparent.

It is very important to note that the derivative vector in general will have a *different direction* from the given vector (see p. 57).

We next consider a very useful question, namely, given the projections of a vector on the coördinate axes, supposed rectangular, to find the projections upon any given line. We consider the problem for vectors in a plane, where we shall need it most, and

leave the analogous considerations in space to the student to work out, or look up, as occasion arises. Let W denote a vector and l a direction, e.g. the direction of the positive y-axis. We denote by (l, W) the angle between the direction l and W measured from l

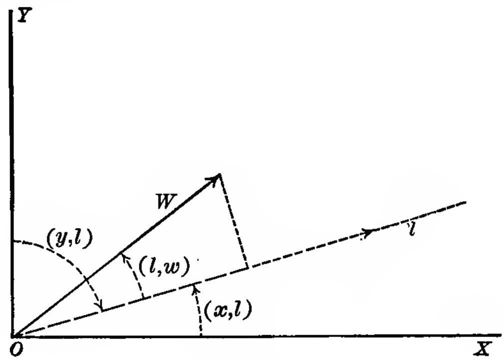


FIG. 4

toward W, and similarly by (x, l) and (y, l) the angle the line l makes with the axes measured from the axes to the line. Then, according to the definition of projection, the projection of W upon l is

$$\begin{aligned} w_l &= w \cos(l, W) = w \cos[(x, W) - (x, l)] \\ &= w[\cos(x, W)\cos(x, l) + \sin(x, W)\sin(x, l)] \\ &= w\cos(x, W)\cdot\cos(x, l) + w\cos(y, W)\cdot\cos(y, l). \end{aligned}$$

But $w\cos(x, W) = w_x$, and $w\cos(y, W) = w_y$. Hence we have

IV. $\boldsymbol{w_l = w_x \cos(x, l) + w_y \cos(y, l)}$, *that is, the projection of a vector upon any line is the sum of the products of each projection upon an axis by the corresponding direction cosine of the line.*

6. The fundamental relation between acceleration and force vectors. With the above information concerning vectors, we may proceed to consider motion other than in a straight line (cf. § 3). Velocities, accelerations, and forces will then be vectors, and the fundamental equation $\boldsymbol{F = mA}$ is to be understood: *the vectors F and A have the same direction, and the magnitude of F is m times the magnitude of A.*

Ex. Show that the above statement is equivalent to: the projection of F upon any line is m times the projection upon the same line of A.

II. PROBLEMS ON VECTORS

The following problems, as well as most of those in the book, should be worked with the aid of a careful figure. Measurement of the figure will then serve as an approximate check on the work. In case of vectors in space the student can at least lay off the projections on straight lines.

1. A sailboat is sailing "into the wind," its course making an angle of 35° with the wind. The speed of the boat is 6 mi. per hr. and that of the wind is 20 mi. per hr. Find the speed and direction of the wind as it appears to a person on the boat.

Solution. The forward velocity of the boat gives a relative backward component to the velocity of the wind. Our problem is to solve the triangle BAD, given $AD = 20$, $BA = 6$, and $\measuredangle BAD = 145°$.

$$\begin{aligned} BD^2 &= 6^2 + 20^2 + 2 \times 6 \times 20 \times \cos 35° \\ &= 436 + 240 \times .819 \\ &= 25.2, \text{ about.} \end{aligned}$$

$$\begin{aligned} \sin DBA &= (20 \sin 145°)/25.2 \\ &= 20 \times .574/25.2 \\ &= .456. \\ DBA &= 27.1°, \text{ about.} \end{aligned}$$

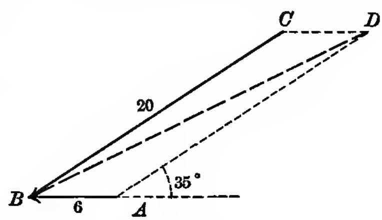


FIG. 5

Hence the apparent speed of the wind is 25.2 mi. an hour, and it apparently makes an angle 27.1° with the course of the boat.

In some problems the formulas of the exercises in this chapter may prove useful, but the student had better rely upon his knowledge of trigonometry and his own ingenuity.

2. Two forces of 12 and 16 lb. respectively act at an angle of 90°. Find magnitude and direction of the resultant.

3. Find the resultant of two forces of the same magnitude f acting at an angle of 45°.

4. Two men kick a football at the same instant. One kicks eastward at the rate of 71 ft. per sec., and the other northwest at the rate of 48 ft. per sec. Find the initial direction and the velocity of the ball.

5. The resultant of two forces is 9. One of them is 3 and the angle which it makes with the resultant is 60°. What is the magnitude of the other force, and what the angle between it and the resultant?

6. Two forces are inclined to their resultant at angles of 120°. How are the magnitudes of the forces related?

7. A balloon rises 1120 ft. per min. while the wind blows it horizontally 370 ft. per min. What is its velocity, and in which direction does it rise? (Use table of natural functions.)

8. In the following table the last three columns are for the projections of the resultant of the forces whose projections are given in the preceding columns. Fill in the blanks.

	X_1	Y_1	Z_1	X_2	Y_2	Z_2	X_3	Y_3	Z_3	X_4	Y_4	Z_4	X_5	Y_3	Z_5	X	Y	Z
a	2	-1	3	4	-6	-2	7	13	2	8	1	3	-5	-1	0			
b	3	6	9				0	0	0	0	0	0	0	0	0	2	7	-2
c	1.7	-2.1	6.4	3	-2.9	2.2	9.3	2.5	6.6	17.24	6.5	-2.8	-20	-8	-12			
d	$\sqrt{3}$	2	$\sqrt{5}$	$1+\frac{1}{\sqrt{3}}$	$-\sqrt{2}$	$\frac{2}{\sqrt{5}}$	$1-\frac{1}{\sqrt{3}}$	$2\sqrt{2}$	-3	$-2\sqrt{3}$	0	0	$\frac{3}{\sqrt{3}}$	5	-1			
e	a	b	$2c$	b	a	0	$-c$	$-c$	0	0	$2c$	$-c$				$a+b$	$a+b$	$a+c$
f	$a-b$	$b-c$	$c-a$	$b-c$	$c-a$	$a-b$	$c-a$	$a-b$	$b-c$	0	0	0	0	0	0			
g	a	0	0	$-\frac{1}{2}a$	$\frac{\sqrt{3}a}{2}$	0	$-\frac{1}{2}a$	$-\frac{\sqrt{3}a}{2}$	0	0	0	0	0	0	0			
h	0	$\cos\frac{2}{5}\pi$	$\sin\frac{2}{5}\pi$	0	$\cos\frac{4}{5}\pi$	$\sin\frac{4}{5}\pi$	0	$\cos\frac{6}{5}\pi$	$\sin\frac{6}{5}\pi$	0	$\cos\frac{8}{5}\pi$	$\sin\frac{8}{5}\pi$	0	1	0			

9. In the following table the last column is for the magnitude and direction cosines of the resultant of the forces whose magnitudes and direction cosines are given in the preceding columns. Fill in the blanks.

	f_1	$\cos\alpha_1$	$\cos\beta_1$	$\cos\gamma_1$	f_2	$\cos\alpha_2$	$\cos\beta_2$	$\cos\gamma_2$	f_3	$\cos\alpha_3$	$\cos\beta_3$	$\cos\gamma_3$	f_4	$\cos\alpha_4$	$\cos\beta_4$	$\cos\gamma_4$	f	$\cos\alpha$	$\cos\beta$	$\cos\gamma$
a	$\sqrt{3}$	1	0	0	2	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	2	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$				
b	10	$\cos 60°$	$\cos 30°$	0	10	$\cos 60°$	$\cos 150°$	0	10	-1	0	0	0	a	b	c				
c	5	1	0	0	5	$-\frac{1}{3}$	0	$\frac{2\sqrt{2}}{3}$	5	$-\frac{1}{3}$	$-\frac{1}{3}\sqrt{6}$	$-\frac{1}{3}\sqrt{2}$	5	$-\frac{1}{3}$	$\frac{1}{3}\sqrt{6}$	$-\frac{1}{3}\sqrt{2}$				
d	12	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	14	$-\frac{6}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$	33	$\frac{2}{11}$	$-\frac{6}{11}$	$\frac{9}{11}$	10	$\frac{3}{5}$	$\frac{4}{5}$	0				
e	4	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	5	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{2}{3}$	6	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$					10	1	0	0
f	18	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	24	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$					0	l	m	n	49	$\frac{6}{7}$	$-\frac{2}{7}$	$\frac{3}{7}$

10. A man jumps from a train running at 10 mi. per hr. in a direction making an angle of 30° with the train and a velocity of 9 ft. per sec. What is his velocity in magnitude and direction relative to the ground?

11. A stream has a current speed a, and a man can row his boat with a speed b. In what direction is he to row if he is to land at a point directly opposite his starting point? In what direction must he row in order to cross in the shortest time? (Consider $a \lesseqgtr b$.)

12. A train is traveling at the rate of 20 mi. per hr., and rain falls vertically with a velocity of 22 ft. per sec. Find the direction of the splashes on the windows.

13. A river $\frac{3}{4}$ mi. wide has a current of 3 mi. per hr.; a ferryboat can cross in 4.5 min. At what angle must it be headed upstream in order to go straight across?

ANALYSIS OF CHAPTER I *

1. Subject-matter of mechanics.

2. British and French units of space, mass, time, velocity, acceleration, and force.

3. Vectors in space and in the plane, and how they are fixed.

4. Vector addition and resolution. Some theorems connecting operations upon vectors with operations upon their projections.

5. The fundamental relation between force and acceleration.

* The analyses of the chapters are intended to point out the more important notions derived from the text, so that the student may have some idea of the results he should have from his study. He should be able to give clear and accurate discussions of each topic noted. He should, moreover, have a clear notion of how to attack each of the problems, whether or not he actually works them all. A good percentage of them should be completed.

CHAPTER II

STATICS

7. Statics considers primarily the question, *When can bodies remain unmoved under the action of forces?* although it will be found that the necessary conditions for a state of rest also allow of certain motions, as uniform motion in a straight line, and for this reason such motions are sometimes considered in statics. We shall here, however, think of our problem as that of finding the conditions for rest. When these are fulfilled the forces are said to be in *equilibrium.* An essential distinction arises on the basis as to whether the forces all act at one point or not, and we take up first the simpler case in which they do, that is, in which they are *concurrent.*

8. Equilibrium under the action of concurrent forces. We shall be concerned only with *rigid bodies,* that is, bodies in which the forces acting produce no measurable change in size or shape.* The forces acting on such a body may evidently be displaced along their lines of action by any amount, for it is clear that the intervening little bars of matter may be regarded as transmitters of the force. For our present purposes, therefore, "concurrent forces" means forces whose lines of action pass through a common point, and we may confine our attention to that point.

From the fundamental law of § 6 we see that, inasmuch as for rest $A = 0$, it follows that $F = 0$, it being of course understood that F comprises all the forces acting on the body, or that it is their resultant. We then have the result

A body can be at rest under the action of concurrent forces when, and only when, their resultant vanishes.

* Of course no *absolutely* rigid bodies exist in nature. Metals which are usually thought of as rigid become quite pliable when made in long thin pieces, like wires. However, the *notion* of rigid bodies is extremely useful in furnishing a simple approximation to what occurs in nature (see also p. 26).

If a number of vectors $V_1, V_2, \cdots, V_n$, are added by putting the initial point of each upon the terminal point of the preceding (cf. Ex. 3, § 5), a polygonal line is formed which is called the *vector polygon*, or in particular, if the vectors are forces, the *force polygon*. The resultant force is the one which, starting from the initial point of the first force, closes the polygon.

Ex. Show that if we add the n vectors $V_1, V_2, \cdots, V_n$ in various orders, we obtain $n!$ force polygons, all of which have the same closing side.

Thus we may state: *A body can be at rest under the action of concurrent forces when, and only when, the force polygon is closed.* For then, and then only, will the resultant be zero. Geometrically, this is the method usually used in treating problems of equilibrium. The analytic statement of the condition is found by projecting the resultant upon the axes and referring to Ex. 10, § 5. If X, Y, and Z denote the projections of the force F upon the axes, the conditions for equilibrium of concurrent forces takes the form

$$\sum_{i=1}^{n} X_i = 0, \qquad \sum_{i=1}^{n} Y_i = 0, \qquad \sum_{i=1}^{n} Z_i = 0. \tag{1}$$

III. PROBLEMS ON EQUILIBRIUM OF CONCURRENT FORCES

[Accompany each problem with a diagram. Each problem should be solved geometrically *and* analytically when possible.]

1. Show that the forces represented by the medians of a triangle are in equilibrium, the positive sense of each median being toward the vertex. The solution follows as an example of the method of attacking these problems.

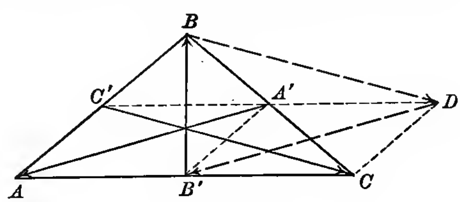


FIG. 6

(a) *Geometric proof.* As the medians meet in a point, the forces are concurrent. We shall show that the force polygon, in this case a triangle, is closed. To do this we shift the force $C'C$ parallel to itself, so that its initial point C' falls upon the terminal point B of $B'B$. Then $C'BDC$ is a parallelogram and $CD = C'B = AC'$, C' being the mid-point of AB. To show the force polygon is closed we merely need to show that DB' is parallel with and equal to $A'A$. This will follow if we can show the two parallelograms $AC'A'B'$ and

$B'A'DC$, of which $A'A$ and DB' are corresponding diagonals, to be equal. But CD is equal to and parallel with AC', and so is $B'A'$, for the segment joining the mid-points of two sides of a triangle is parallel with the third side and equal to one half of it. Finally $AB' = B'C$, and hence the parallelograms are equal and the theorem is proven.

(b) *Analytic proof.* If the coördinates of A, B, and C are respectively (x_1, y_1), (x_2, y_2), (x_3, y_3), the coördinates of A' are $\frac{1}{2}(x_2 + x_3)$, $\frac{1}{2}(y_2 + y_3)$, of B', $\frac{1}{2}(x_3 + x_1)$, $\frac{1}{2}(y_3 + y_1)$, of C', $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$. The components or projections of the forces $A'A$, $B'B$, and $C'C$ on the x-axis are therefore $X_1 = x_1 - \frac{1}{2}(x_2 + x_3)$, $X_2 = x_2 - \frac{1}{2}(x_3 + x_1)$, $X_3 = x_3 - \frac{1}{2}(x_1 + x_2)$. Hence $X_1 + X_2 + X_3 = x_1 + x_2 + x_3 - \frac{1}{2}(2x_1 + 2x_2 + 2x_3) = 0$. Similarly, $Y_1 + Y_2 + Y_3 = 0$, and the forces, having a vanishing resultant, are in equilibrium. In this problem the analytic treatment is briefer and more elegant, though this is by no means always the case.

2. Prove that three forces of magnitude 5, 6, and 12 can never be in equilibrium.

3. Let AB and CD be diameters of a circle. Three concurrent forces are represented in magnitude and direction by AB, DC, and $2BD$. Show they are in equilibrium.

4. Three concurrent forces are determined by their magnitudes and the angles their directions make with the axes as follows:

$$\begin{array}{lllll} F_1; & f = 20, & \alpha = 30°, & \beta = 60°, & \gamma = 90°. \\ F_2; & f = 14, & \alpha = 90°, & \beta = 45°, & \gamma = 45°. \\ F_3; & f = 10, & \alpha = 45°, & \beta = 45°, & \gamma = 90°. \end{array}$$

Find the force which will hold them in equilibrium.

5. Find the resultant of the forces given by their projections upon the axes:

$$\begin{array}{llll} F_1; & f_x = 5, & f_y = 7, & f_z = 11. \\ F_2; & f_x = 6, & f_y = -5, & f_z = 2. \\ F_3; & f_x = 9, & f_y = -2, & f_z = -6. \end{array}$$

6. Three concurrent forces are in equilibrium. Show they lie in a plane. Show further that if their magnitudes are f_1, f_2, f_3, and the angles between them are α_{23}, α_{31}, and α_{12}, then

$$\frac{f_1}{\sin\alpha_{23}} = \frac{f_2}{\sin\alpha_{31}} = \frac{f_3}{\sin\alpha_{12}}. \qquad \text{(Lamé's Theorem)}$$

Hint. Consider the force triangle.

7. Let a system of forces be represented by OA, OB, OC, $\cdots$, ON. Show that if they are in equilibrium, the coördinates of O are the averages of the corresponding coördinates of A, B, C, $\cdots$, N; in other words, O is the center of mass of a system of equal particles at the points $A, B, C, \cdots, N$ (see p. 27).

8. A weight W upon an inclined plane whose angle with the horizon is i, may be held at rest either by a force Q acting up the incline, or by a force P acting horizontally. Show that $W = PQ/\sqrt{P^2 + Q^2}$, and $\cos i = Q/P$. Is your result in poundals or pounds?

9. Find the angles between three forces $2P$, $3P$, and $4P$ which are in equilibrium.

10. Two rafters meeting at an angle 60° in a vertical plane support a chandelier weighing 90 lb. What is the force along each rafter?

11. A body of mass 140 lb. is attached to the ends of two ropes of length 6 and 8 ft. respectively, which are fastened to a horizontal beam at two points 10 ft. apart. Find the tensions on the ropes.

12. A bar weighing 100 lb. is suspended by chains passing from a ring to its ends. The chains make an angle of 45° with the vertical. Find the tension on the chains.

Hint. Consider the bar replaced by two weights of 50 lb. each at its end points and connected by a weightless rigid bar.

13. A body is kept at rest on a smooth inclined plane by two forces, each equal to half the weight of the body, the one acting horizontally and the other directly up the plane. Find the angle of inclination of the plane.

14. Two smooth rectangles have a horizontal edge in common, their faces being inclined to a horizontal plane at angles a_1 and a_2 respectively. Weights w_1 and w_2 rest one on each plane, and are connected by a string running smoothly over the common horizontal edge. If the system is in equilibrium, find the ratio of w_1 to w_2.

Hint. Consider, say, the weight w_1, remembering that the string transmits the force exerted upon w_1 by w_2 in line with the plane on which w_1 rests.

Friction. When two bodies are in contact along plane surfaces, there is usually a pressure keeping them together and causing more or less of an interplay of the slight roughness of their surfaces. There thus arises a certain resistance to any force tending to make the surfaces slide over one another. This resistance is a force exerted by the protruding particles against each other, and has a direction opposite to that of the attempted motion and is called a frictional force. As long as there is no motion, the frictional force exactly balances the force tending to produce motion. But it is found that when the moving force exceeds a certain limit, the body does move, although still *retarded* by the frictional force. Experiments justify as an approximation the assumption that this force is proportional to the force pressing the two bodies together, that is, to the normal component n of the resultant of the forces acting at the common surface. We have then the result

(1) The frictional force has a direction opposite to that of the resultant of the other forces.

(2) It has a magnitude f which,

(a) when no motion takes place, is equal to the magnitude of this resultant;

(b) when motion takes place, is proportional to the magnitude of the normal component of their resultant. In symbols $f = \mu n$, where μ is a

proportionality factor, and is called the *coefficient of friction*. It depends upon the character of the surfaces in contact. The student should avoid feeling that the above is more than an approximation, and apply it only in the case of bodies sliding on one another. There are many other kinds of frictional forces, too numerous and varied to receive attention here.

15. Determine the frictional force which keeps a body of weight 1 lb. at rest on a plane inclined at an angle of 30°, the slipping point being not yet reached.

16. If a plane be tilted farther and farther, the normal component of the gravitational force acting upon a body on the surface of the plane diminishes, while the component tending to produce slipping increases. The angle of inclination e at which the slipping begins, is called the *angle of friction*. Prove $\mu = \tan e$.

17. Suppose a body rest on a table and a force of magnitude h tends to move it. The magnitudes of the friction f depend upon h. Draw the graph of f as a function of h up to and beyond the slipping point. What is the value of h at the slipping point?

18. Let a be the angle of inclination of a plane, and f the frictional force tending to prevent or retard a body's sliding along the plane. Draw the graph of f as a function of a up to and beyond the angle of friction. Take e, say at 20°.

19. A mass of 10 lb. rests upon a table and can be just moved by a force of 3 lb. acting horizontally. Find the coefficient of friction and the direction and magnitude of the resultant reaction of the plane.

20. A body of weight w rests on a rough horizontal table. If a force be applied with an upward component, this upward component will lessen the normal reaction and hence the friction. Show that the least force which will move the body makes an angle e with the horizontal and has a magnitude $w \sin e$.

21. Two weights of 10 and 20 lb. lie upon a rough inclined plane connected by a string which passes around a pulley in the plane. Find the greatest inclination of the plane consistent with equilibrium of the system, the coefficient of friction being μ.

22. How high can a particle rest in a hemispherical basin of radius r, the coefficient of friction being μ?

9. Nonconcurrent forces; moments. We now turn to the consideration of a system of forces acting at various points of a rigid body whose lines of action do not all intersect in one point. Besides a tendency to translate there will, in general, be a tendency to rotate the body about some line in it. If there is to be equilibrium, the tendency to rotate about any line whatever in the body

must be zero. We consider the question of measuring this tendency to rotate.

Let us first examine a force whose line of action is perpendicular to the axis of rotation.

The angle between two nonintersecting lines is defined to be the angle between two intersecting lines which are parallel, one with each of the given lines.

The tendency of such a force to turn about the axis is found to be proportional to the magnitude f of the force and to the distance p of its line of action from the axis, or the *arm* of the force as it is called; in other words, the tendency is proportional to $f \cdot p$.

The student will find the best substantiation of this empirical fact in the well-known properties of levers and balances. (See Mach, *The Science of Mechanics*, translated by McCormack, Chicago, 1902.)

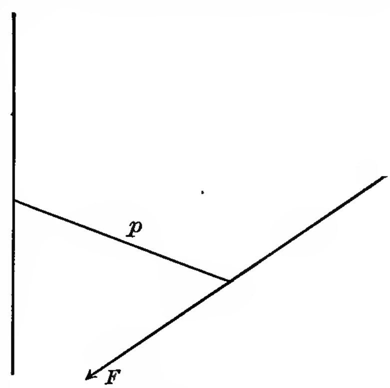


Fig. 7

This quantity $f \cdot p$ is called the *moment* of the force about the axis. If a number of forces are acting, some tending to turn in one direction and others in the opposite, we fix on a positive direction and give a positive sign to the moments of the forces tending to turn in that direction, and a negative sign to the moments of those tending to turn in the opposite direction.

Should the force F not be perpendicular to the axis, we resolve it into two components, — F_a parallel with the axis, and F_n perpendicular to the axis (see Fig. 8).

The component F_a evidently has no tendency to turn about a line parallel to it, so that the turning effect of F is that of F_n. We therefore define the moment in this case to be $f_n \cdot p$, where f_n is the magnitude of F_n and p the distance of its line from the axis.

When the forces studied all lie in one plane perpendicular to the axis the problem is usually regarded as a *plane problem,* and we speak of the moments of the forces about a *point,* namely, the point in which the axis pierces the plane. The following exercises are important parts of the theory of moments and should be thoroughly studied.

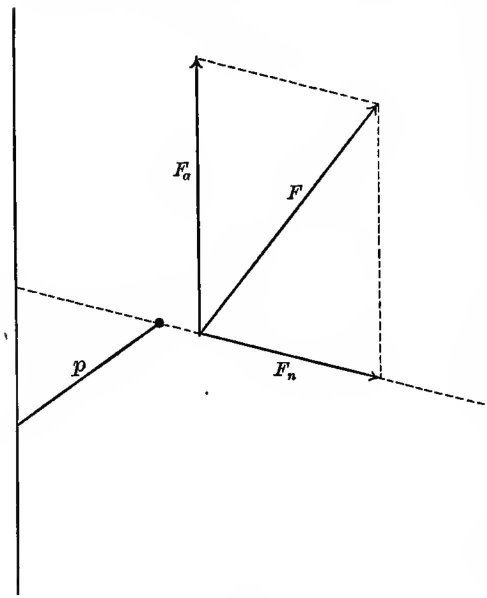


FIG. 8

Ex. 1. Let P be any point upon a force vector F or upon its line of action. Prove that *the moment of F about a point O is OP times the projection $f_{p'}$ upon a line perpendicular to OP.*

Hint. Use the previous definition, and similarly of triangles.

Ex. 2. Let OX be an axis, OP a perpendicular dropped upon it from any point P of a force vector F or of its line of action, and let $f_{p'}$ be the projection of F on a line perpendicular to the plane of OX and OP. Calling $OP = p'$, show that *the moment of F* about OX is $p'f_{p'}$.

Ex. 3. From the preceding, together with I of § 5, prove that *the moment of the resultant of several concurrent forces is the sum of the moments of the separate forces.*

Hint. Take the common point of the lines of force for P.

Ex. 4. Prove that if the moment of F about OX vanishes, F and OX lie in the same plane.

10. Couples. If two equal and opposite forces are concurrent, their effect is nil. If not concurrent, they are said to form a *couple.* A couple has no tendency to translate the body as a whole, as may be seen by putting the body on a track running in any given direction. Equal forces will be exerted on the body in opposite directions. The couple will, however, have a tendency to rotate

the body. This tendency is measured by the so-called *moment of the couple,* which is simply the sum of the moments of its two forces about a point midway between their lines of action.

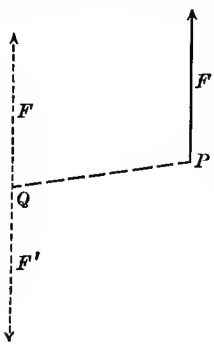


FIG. 9

Ex. 1. Prove that *the moment of a couple about all points of its plane is the same, and equal to $p \cdot f$, where p is the distance between the parallel forces and f is their common magnitude.*

Ex. 2. Prove that the moments of a couple about all axes which have the same direction is the same, and that this common value is *the moment of one of the forces about an axis with the given direction through the other force.*

The point of application of a force F may be thought of as changed from P to Q, provided at the same time a couple be introduced consisting of a force F at P and a force at Q equal to F in magnitude but opposite in direction. For the new system is merely the original force with two mutually annulling forces introduced. It follows that any system of nonconcurrent forces may be replaced by a system that is concurrent at any given point O together with a system of couples. Thus the effect of a system of nonconcurrent forces is the same as that of

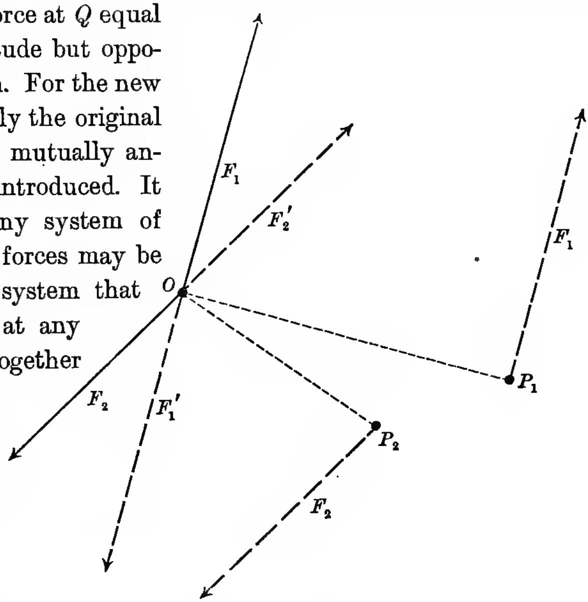


FIG. 10

(1) *a single force applied at any given point O and equal to the resultant the given forces would have if concurrent, and*

(2) *a set of couples, one for each force of the given system, and each having for its moment about any axis the moment of the corresponding force about a line through O parallel to the axis* (see Ex. 2).

The moments of the couples about an axis through O will, in general, vary with O, because to change the point of application O of the resultant means to introduce a new couple. Only in case the resultant vanishes will this new couple vanish, and the sum of the moments of the couples be independent of the point O chosen.

11. Equilibrium. A set of forces is said to be in *equilibrium* if a resting body remains at rest under their action. Having studied the tendencies of forces to produce motion, we may state the following necessary and sufficient conditions for equilibrium

(1) *the resultant of all the forces, regarded as concurrent, vanishes;*

(2) *the sum of the moments of the forces about any axis vanishes.*

Our next task is to express these conditions analytically. If the forces $F_1, F_2, \cdots, F_n$ be projected on to the three coördinate axes, giving $f_{1x}, f_{2x}, \cdots, f_{nx}$; $f_{1y}, f_{2y}, \cdots, f_{ny}$; $f_{1z}, f_{2z}, \cdots, f_{nz}$, we have, referring to I, § 5,

$$\left.\begin{array}{l} f_{1x}+f_{2x}+\cdots+f_{nx}=0 \\ f_{1y}+f_{2y}+\cdots+f_{ny}=0 \\ f_{1z}+f_{2z}+\cdots+f_{nz}=0 \end{array}\right\}, \quad \text{or} \quad \left.\begin{array}{l} \sum_i f_{ix}=0 \\ \sum_i f_{iy}=0 \\ \sum_i f_{iz}=0 \end{array}\right\}. \qquad \text{I}$$

These conditions express the demand that there shall be no tendency to translate the body as a whole. If they are satisfied, the forces may be grouped in couples whose effect to rotate about an axis remains the same, no matter how the axis is shifted parallel to itself. We may therefore take it through the origin. If the sum of the moments of the couples, or, what is the same thing, of the original force system is to vanish when taken about any axis, it must, in particular, when taken about the three coördinate axes.

But, conversely, if the sum of the moments about each coördinate axis vanishes, it vanishes about every axis.

Ex. Prove this. Use Ex. 4, § 9.

Hence we have the second condition in the form: *the moments of the forces about the three coördinate axes must vanish.* We

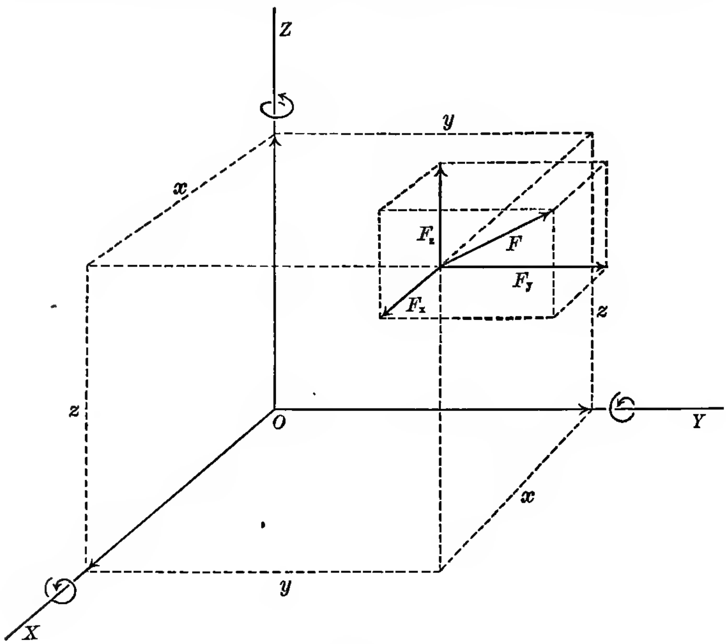


FIG. 11

proceed now to calculate analytically the sums of the moments about the axes. Let us agree to call a positive rotation about an axis one which would drive a right-handed screw forward in the direction of the axis, as denoted in the accompanying figure. Each force is split into its components, and the moments of each component about the axes are tabulated in the above scheme,

Axis	Component		
	F_x	F_y	F_z
OX	0	$-zf_y$	$+yf_z$
OY	$+zf_x$	0	$-xf_z$
OZ	$-yf_x$	$+xf_y$	0

(x, y, z) being the coördinates of the point of application of the force.

We have, therefore, writing down these moments for all the forces and forming the sums, the second set of conditions for equilibrium

$$(y_1 f_{1z} - z_1 f_{1y}) + (y_2 f_{2z} - z_2 f_{2y}) + \cdots + (y_n f_{nz} - z_n f_{ny}) = 0,$$
$$(z_1 f_{1x} - x_1 f_{1z}) + (z_2 f_{2x} - x_2 f_{2z}) + \cdots + (z_n f_{nx} - x_n f_{nz}) = 0,$$
$$(x_1 f_{1y} - y_1 f_{1x}) + (x_2 f_{2y} - y_2 f_{2x}) + \cdots + (x_n f_{ny} - y_n f_{nx}) = 0,$$

or

$$\left\{\begin{aligned} \sum_i (y_i f_{iz} - z_i f_{iy}) &= 0 \\ \sum_i (z_i f_{ix} - x_i f_{iz}) &= 0 \\ \sum_i (x_i f_{iy} - y_i f_{ix}) &= 0 \end{aligned}\right\}. \quad \text{II}$$

Ex. 1. Show that the moment of a force about a point on its line of action vanishes; also about any axis intersecting its line of action or parallel to it.

Ex. 2. Show that the moment about the origin of the force F with projections f_x and f_y on the axes and with point of application (x, y) is $xf_y - yf_x$ as follows: find the equation of the line of action, find its distance from the origin, and multiply by the magnitude f of the force.

Ex. 3. Show that if three forces acting on a rigid body are in equilibrium (a) they lie in a plane; (b) they are either concurrent or parallel; and (c) the force triangle is closed, the relation between the forces being given by Lamé's theorem (cf. problem 6, § 8).

Ex. 4. *The composition of parallel forces.* The *resultant* has been defined only for concurrent forces. A natural extension of the idea to parallel forces is: the resultant of a set of parallel forces is the force which would hold the set in equilibrium, *with its direction reversed.* By the theory of moments show that the resultant of two parallel forces F_1 and F_2, with magnitudes f_1 and f_2, has a magnitude $f_1 + f_2$ or $|f_1 - f_2|$ (the bars denoting the absolute value of the difference), according as the directions of the forces are the same or opposite, and that the point of application of the resultant divides the line joining the points of application in the ratio $f_2 : f_1$, internally or externally, according as the forces are similarly or oppositely directed.

Ex. 5. Generalize to three parallel forces not all in the same plane. Consider the points where their lines cut a perpendicular plane, and let their coördinates be (x_1, y_1), (x_2, y_2), (x_3, y_3). Find first the point of application of the resultant of two of the forces and then bring in the third.

IV. PROBLEMS ON MOMENTS AND THE EQUILIBRIUM OF FORCES

[In solving problems in equilibrium of forces the student should accompany each problem by a carefully drawn diagram in which are marked *all* forces acting on the body, or *on each separate body* in case more than one is involved, being careful to include "reactions" (pressures or tensions) at all points where bodies are in contact, tensions of strings, and so forth. Then he should apply the conditions for equilibrium to the body, or, in case of several, to each body and to the system as a whole. Geometric relations in the diagram should also be written down. The result will be a system of equations to be solved for the unknown forces. The number of equations should agree with the number of unknowns if the problem is determinate. Notice that to a force in a plane correspond two unknowns, and in space three, though this number is diminished when either the magnitude or direction or other similar data is given with respect to the force. In the following problems the forces all lie in one plane, and moments need only be taken with respect to one point.]

1. A uniform rod AB of length $2l$ and weight w rests with the end A against a smooth vertical wall, while to the lower end B is fastened a string BC of length $2b$, coming from a point C in the wall directly above A. If the system is in equilibrium, determine the angle $ACB = \theta$. Show that the tension on the string $T = w \cdot \sec\theta$, and that the pressure against the wall $P = w \cdot \tan\theta$.

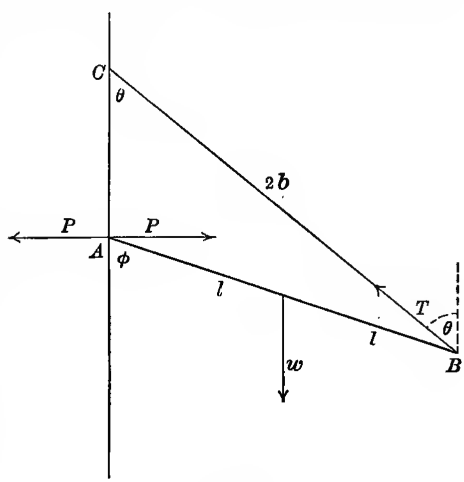


Fig. 12

Solution. We consider the forces acting on the bar. As the wall is "smooth," the pressure between the bar and the wall is perpendicular to the wall, for otherwise there would be slipping. The weight w may be considered as acting directly downward at the mid-point of the bar, and the tension T acts along the string. Applying the first conditions of equilibrium, the sum of the horizontal components must vanish:

$$P - T \sin\theta = 0; \tag{1}$$

and also the sum of the vertical components:

$$-w + T\cos\theta = 0. \tag{2}$$

Also, taking moments about A,

$$-w \times l \sin \phi + T \times 2l \sin (ABC) = 0. \tag{3}$$

The unknowns are θ, P, T, and ϕ and (ABC). We therefore need two more relations. These relations are geometric in character, and are

$$(ABC) = \phi - \theta, \tag{4}$$

and, from the law of sines,
$$\frac{\sin \phi}{2b} = \frac{\sin \theta}{2l}. \tag{5}$$

We now proceed to answer the questions proposed. Equation (2) solved for T gives $T = w \sec \theta$, one of the required results. This value substituted in (1) gives $P = w \tan \theta$, a second required result. To determine θ we substitute the value of T, and the value of ABC given by (4) in equation (3), at the same time dividing by wl and expanding $\sin(\phi - \theta)$:

$$-\sin \phi + 2 \sec \theta (\sin \phi \cos \theta - \cos \phi \sin \theta) = 0,$$

or
$$-\sin \phi + 2 \sin \phi - 2 \cos \phi \tan \theta = 0,$$

that is,
$$\tan \phi = 2 \tan \theta.$$

Equation (5) may be written $\sin \phi = \frac{b}{l} \sin \theta.$

Unless $\theta = 0$, we may divide this equation by the preceding and obtain

$$\cos \phi = \frac{b}{2l} \cos \theta.$$

Squaring and adding, we have

$$1 = \frac{b^2}{4l^2}(4 \sin^2\theta + \cos^2\theta) = \frac{b^2}{4l^2}(1 + 3 \sin^2\theta),$$

whence
$$\sin \theta = \pm \sqrt{\frac{4l^2 - b^2}{3b^2}}.$$

The excluded value $\theta = 0$ also gives a position of equilibrium, so that the final required result is

$$\theta = 0, \quad \text{or} \quad \theta = \text{arc} \sin \sqrt{\frac{4l^2 - b^2}{3b^2}}.$$

Remarks. (a) It is not necessary that the components of the forces along *horizontal* and *vertical* lines be taken in the first conditions of equilibrium. Any two intersecting lines will do. In this problem horizontal and vertical directions were chosen because two of the forces had those directions.

(b) Moments might have been taken about any point. A was chosen because thereby the unknown force P was eliminated from the equation.

(c) In choosing the geometric relations we did, we were guided by the necessity of having two more relations in ϕ and (ABC) in order to eliminate these unknowns from equation (3).

(d) It is interesting to ask whether the equilibrium is *possible*. This is the case if a real value of θ has been obtained, for then all the conditions are satisfied. In order that θ be real the expression under the radical sign must be positive and less than 1. This will be the case if b lies between l

and $2l$; that is, the string must be longer than the bar but less than twice its length.

2. Weights of 1, 2, 3, 4, and 5 lb. act on a bar at distances 1, 2, 3, 4, and 5 ft. from one end. Find the magnitude and point of application of the force which will hold the system in equilibrium (neglecting the weight of the bar).

3. A bar of uniform thickness weighs 10 lb. and is 5 ft. long. Weights of 9 and 5 lb. are hung from its extremities. On what point will it balance? (Assume the weight of the bar to act downward from its center).

4. Find the true weight of a body which weighs 8 oz. and 9 oz. in the right and left pans of a false balance. (The balance is "false" because the two arms of the beam are of unequal length.)

5. A span of a bridge 40 ft. long weighs 10 T. and is supported by two similar piers at the ends. What is the thrust on each pier if a wagon weighing 2 T. is (a) at the middle of the span? (b) two thirds the way across?

6. A uniform bar 2 ft. 8 in. long weighing $5\frac{1}{4}$ lb. is supported by a smooth peg at one end and by a vertical string 4 in. from the other end. Find the tension of the string, the reaction of the peg, and the inclination of the bar. (Note that when anything is *smooth*, or frictionless, the force is *normal* to the surfaces in contact.)

7. A rope of length l is tied to a column and a man is pulling at the other end. If he exerts a force independent of the direction of the rope, at what point of the column should the rope be attached in order that the man's efforts be most effective in overturning the column? (Assume that the man's hands are on a level with the base of the column.)

8. A gate is hung in the usual manner by two hinges on a gate post. Indicate the forces acting on the gate when it hangs open and in equilibrium, and show how it may happen that the reaction on one of the hinges is wholly horizontal.

9. A cylinder of length $2l$ and radius r rests with a point of one base on a rough floor and with a point of the other base against a smooth wall, so that its axis lies in a plane perpendicular to the floor and wall and makes an angle i with the floor. Find the frictional force on the floor which keeps it in this position, and show that it vanishes when $\tan i = l/r$. What does this result mean with respect to the center of the cylinder?

10. A uniform rod is suspended from a hook by two strings of length a and b tied to its ends. Show that in a position of equilibrium the tensions on the strings are proportional to a and b.

11. A weightless rod AB of length l can turn freely in a vertical plane about A. A weight w is suspended from a point C of the rod distant c from A. A string attached at B and making an angle 150° with the rod holds it in equilibrium in a horizontal position. Find the tension on the string and the magnitude of the pressure at A.

12. A uniform thin rough rod passes under one peg and over a second higher one, its center being above the higher peg and distant from the higher and lower pegs a and b respectively. Show that if the rod is upon the point of motion, the coefficient of friction is $\mu = (b - a)\tan i/(b + a)$, where i is the inclination of the rod.

For further problems in equilibrium the student is recommended to study the theory of balances and scales, such as freight and postal scales, also derricks and hoisting devices. Examples of the kind are not given here because of the space necessary to describe each apparatus. Moreover, it will be of great value to the student to formulate his own problem.

12. Centers of mass of systems of particles. We shall be concerned with a system of particles connected by weightless bars. A *particle* is the idea derived by imagining all the mass of a body concentrated at a point, whereas a *weightless bar* is to be thought of as a device for keeping the distance between two particles constant. Whereas such things do not exist in nature, problems are simplified by their use, and the results obtained, if not strictly faithful to actuality, are frequently in error by less than the errors of observation; * moreover, they furnish a basis for more rigorous developments. An example of the latter use of such notions appears in the next section.

Near the earth's surface the attraction of gravity furnishes an example of a *set of parallel forces* acting on the above-mentioned system with magnitudes proportional to the masses. We ask, What force will hold this set in equilibrium? Let the particles have masses $m_1, m_2, \cdots, m_n$, and be situated at points (x_1, y_1, z_1), $(x_2, y_2, z_2), \cdots, (x_n, y_n, z_n)$. Call the balancing force F, with projections f_x, f_y, f_z upon the coördinate axes, and with point of application $(\bar{x}, \bar{y}, \bar{z})$. Applying the conditions I of § 11, in which the balancing force is included, we have

$$\left.\begin{aligned} &f_x + 0 + 0 + \cdots + 0 = 0 \\ &f_y + 0 + 0 + \cdots + 0 = 0 \\ &f_z - m_1 g - m_2 g - \cdots - m_n g = 0 \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} &f_x = 0 \\ &f_y = 0 \\ &f_z = g\sum_{i=1}^{n} m_i = Mg \end{aligned}\right\},$$

* As an illustration consider a symmetrical fly wheel with its cylindrical axle, which we shall suppose to protrude more on one side than on the other. The gravitational forces acting on the system may be considered as acting on a particle with the mass of the wheel situated at its geometric center and connected by a weightless bar to a particle with the weight of the axle situated at its geometric center.

where M is the total mass of the system. This gives F by its components. Its point of application is involved in conditions II, into which we introduce the projections of F already determined:

$$\begin{aligned}
&[\bar{y}(Mg)-0]+[y_1(-m_1g)-0]+[y_2(-m_2g)-0]+\cdots=0,\\
&[0-\bar{x}(Mg)]+[0-x_1(-m_1g)]+[0-x_2(-m_2g)]+\cdots=0,\\
&[0-0] \qquad +[0-0] \qquad\qquad +[0-0] \qquad\qquad +\cdots=0,
\end{aligned}$$

or $$\bar{y}=\frac{g\sum y_im_i}{Mg}=\frac{\sum y_im_i}{M}, \quad \text{and} \quad \bar{x}=\frac{\sum x_im_i}{M}.$$

Thus the point of application is not determined, for the conditions of equilibrium will be satisfied for any $\bar{z}$. The explanation is that the point of application of the balancing force may be shifted to any point of its line of action. Suppose we ask whether there is a point at which the body will balance even though turned in a different direction, or, what amounts to the same thing, when the parallel forces on the particles act in a different direction, say opposite to that of the x-axis. We should then have for each mass

$$f_{ix}=-m_ig, \quad f_{iy}=0, \quad f_{iz}=0;$$

and conditions I would give us

$$f_x=Mg, \quad f_y=0, \quad f_z=0;$$

while conditions II would yield

$$\bar{z}=\frac{\sum z_im_i}{M}, \qquad \bar{y}=\frac{\sum y_im_i}{M};$$

it is now $\bar{x}$ undetermined. If, however, we take the point

$$\bar{x}=\frac{\sum x_im_i}{M}, \quad \bar{y}=\frac{\sum y_im_i}{M}, \quad \bar{z}=\frac{\sum z_im_i}{M} \quad \left(M=\sum m_i\right),$$

the body will balance at this point against a system of forces parallel to either of the two axes considered, and, as may easily be proven, *against any set of parallel forces with magnitudes varying as the masses.* This point is called the *center of mass.* (The expressions *centroid* and *center of gravity* are also frequently used, but sometimes with slightly different meanings, so that we shall confine ourselves to the term given.) The center of mass is of

great importance in the mechanics of systems of particles and of bodies.

Ex. Prove the assertion just made, that the system will balance at its center of mass for any direction of the parallel forces. Let the direction cosines of the direction of the forces be $\cos\alpha$, $\cos\beta$, and $\cos\gamma$. Conditions I give the components f_x, f_y, f_z of the balancing force, and in conditions II it will be found that if the values given above for $\bar{x}$, $\bar{y}$, $\bar{z}$ are used, the coefficients of $\cos\alpha, \cos\beta$, and $\cos\gamma$ vanish separately, so that the conditions are fulfilled no matter what these cosines be. Show that the magnitude of the balancing force is Mg.

V. PROBLEMS ON THE CENTER OF MASS

1. A ball of 2 lb. and one of 20 lb. are fixed to the ends of a uniform bar 5 ft. long and of weight 5 lb. Find the center of mass of the system.

2. A fly wheel weighing 2 T. rests upon an axle of 400 lb., 3 ft. long, the plane of the wheel dividing the axle into two parts in the ratio of 1 to 3. Find the center of mass of the system.

3. Given two masses m_1 and m_2 situated at the points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, show by considering moments, that the center of mass divides the segment joining the masses in the ratio of m_2 to m_1. Hence, applying a formula of Analytic Geometry, obtain the formulas for $\bar{x}$, $\bar{y}$, $\bar{z}$ for two masses. A third mass may now be added by considering the first two united in their center of mass, and applying the same method. Generalizing thus, obtain an independent proof of the formulas for $\bar{x}$, $\bar{y}$, $\bar{z}$.

4. Show that the center of mass of three equal particles at the vertices of a triangle is at the intersection of the medians.

5. The vertices of a square carry weights 1, 1, 1, and 2. Determine the position of the center of mass.

13. Centers of mass of continuous bodies. If equal volumes taken from all parts of a body weigh the same, the body is said to have uniform density, and the density is defined to be the constant ratio of mass to volume. If, however, there is no such *constant* ratio, we have recourse to the method of derivatives, saying first, the *average density* of a portion of a body is the mass of that portion divided by its volume; and secondly, the *density at a point* is the limit of the average density of a volume, including the point, as the dimensions of the including volume approach zero. Thus for a nonuniform body the density will usually vary from point

to point, that is, will be a function of x, y, and z, for which we shall use the notation $d = \delta(x, y, z)$.

Turning now to the problem of determining the centers of mass of continuous bodies, we imagine the body split up into cuboids by planes parallel with the coördinate planes. The mass of one of the cuboids will be approximately the volume multiplied by the density $\delta(x, y, z)$, say at the mid-point of the cuboid. We have, therefore, for the x-coördinate of the center of mass of the system of cuboids, according to the preceding paragraph, approximately

$$\bar{x} = \frac{\sum\sum\sum x\delta(x, y, z)\Delta x\Delta y\Delta z}{\sum\sum\sum \delta(x, y, z)\Delta x\Delta y\Delta z},$$

the summations being extended over all cuboids containing masses, and no others. We obtain now the center of mass of the given continuous body by passing to the limit

$$\bar{x} = \frac{\lim\sum\sum\sum x\delta(x, y, z)\Delta x\Delta y\Delta z}{M} = \frac{\iiint x\delta(x, y, z)\,dxdydz}{M},$$

and similarly

$$\bar{y} = \frac{\lim\sum\sum\sum y\delta(x, y, z)\Delta x\Delta y\Delta z}{M} = \frac{\iiint y\delta(x, y, z)\,dxdydz}{M},$$

$$\bar{z} = \frac{\lim\sum\sum\sum z\delta(x, y, z)\Delta x\Delta y\Delta z}{M} = \frac{\iiint z\delta(x, y, z)\,dxdydz}{M},$$

where

$$M = \lim\sum\sum\sum \delta(x, y, z)\Delta x\Delta y\Delta z = \iiint \delta(x, y, z)\,dxdydz$$

is the *total mass of the body*. In all the integrals the limits of integration are determined by the surfaces bounding the body, just as in the volume problems of the Integral Calculus.

Ex. Show that the formulas of §§ 12 and 13 hold also for oblique axes, except that M no longer admits the interpretation of being the mass.

VI. PROBLEMS ON CENTERS OF MASS OF CONTINUOUS BODIES

Unless the contrary is specified, the body is to be assumed homogeneous.

1. Find the center of mass of a hemisphere of radius r whose density varies as the distance from the plane surface.

Solution. Choosing the plane as xy-plane with origin at the center of the sphere, we see at once by symmetry that $\bar{x} = \bar{y} = 0$, whereas by the above formulas, since $\delta(x, y, z) = kz$,

$$z = \frac{k}{M}\int_{-r}^{+r}\int_{-\sqrt{r^2-x^2}}^{+r\sqrt{r^2-x^2}}\int_0^{\sqrt{r^2-x^2-y^2}} z^2 dz dy dx,$$

and
$$M = k\int_{-r}^{+r}\int_{-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}}\int_0^{\sqrt{r^2-x^2-y^2}} z dz dy dx,$$

whence, as the student may verify by evaluating these integrals,

$$M = \frac{k\pi r^4}{8} \quad \text{and} \quad \bar{z} = \frac{k\pi r^5}{15} \div \frac{k\pi r^4}{8} = \frac{8\,r}{15}.$$

2. Determine the center of mass of a homogeneous tetrahedron.

Hint. Oblique axes should be used. The bounding planes may then be written $x = 0$, $y = 0$, $z = 0$, $x/a + y/b + z/c = 1$.

3. Find the center of mass of a homogeneous hemisphere.

4. Find the center of mass of that half of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ which lies to one side of the plane $x = 0$.

5. Find the center of mass of that part of the paraboloid $x^2/a^2 + y^2/b^2 = 2\,z/c$ which lies below $z = h$.

6. Find the center of mass of a plate in the form of a sector of a circle of radius r and angle $2\,a$. In particular, put $a = \pi/2$ and get the centroid of a semicircular plate. For $a = \pi$ the centroid should lie at the geometric center. Check the result in this way.

14. Centers of mass of some bodies of special shapes. The formulas of the preceding sections take on simpler forms in special cases. Some of these we here mention, leaving most of the developments to the student.

(a) *Cylinders,* or bodies bounded by a cylindrical surface and two parallel planes, the density being the same along all lines parallel to the elements of the cylindrical surface. The last means that the density depends, if we take the z-axis in the direction of the elements, on x and y only, i.e. $d = \delta(x, y)$. If the xy-plane be taken halfway between the parallel bases, we see, by symmetry, that $z = 0$, and furthermore we may carry out one integration in the other formulas:

$$\bar{x} = \frac{\iiint_{-\frac{h}{2}}^{\frac{h}{2}} x\delta(x, y)\, dzdxdy}{M}$$

$$= \frac{h\iint x\delta(x, y)\, dxdy}{M},$$

$$M = h\iint \delta(x, y)\, dxdy,$$

and similarly,
$$\bar{y} = \frac{h\iint y\delta(x, y)\, dxdy}{M}.$$

The form of the integrals suggests "the center of mass of an area," and this expression is sometimes used, though its real meaning should be kept in mind. The quantity $h \cdot \delta(x, y)$ is sometimes called the *surface density*. Low cylinders are occasionally called *plates*.

(b) *Straight wires*. Take the x-axis through the wire. Then $\delta(x)$ means the "linear density" (that is, the mass of a piece of the wire divided by its length, or, in case this quotient varies, its limit). The student should show that

$$\bar{x} = \frac{\lim \sum x\delta(x)\Delta x}{\lim \sum \delta(x)\Delta x} = \frac{\int x\delta(x)\, dx}{M},$$

where
$$M = \int \delta(x)\, dx.$$

(c) *Bodies of revolution*, in which the density is the same at all points of any plane perpendicular to the axis of revolution. These are a simple generalization of the preceding, in which $\delta(x)$ means the mass of a plate bounded by two plaues perpendicular to the axis and a distance Δx apart, divided by Δx, or, in case this quantity varies, $\delta(x)$ is its limit as $\Delta x \doteq 0$. In particular, if the volume density is constant, say k, then $\delta(x)$ is k times the area of a cross section a distance x from the origin. The same formulas of course hold.

Example. Consider problem 1 of the preceding paragraph. Taking the x-axis as the axis of revolution of the hemisphere, we have $\delta(x)$ = density times the area of a cross section $= kx(\sqrt{r^2 - x^2})^2$. Whence

$$\bar{x} = \frac{k\int_0^r x^2(r^2 - x^2)\,dx}{k\int_0^r x(r^2 - x^2)\,dx} = \frac{8r}{15}.$$

The student will notice how much more simply the integrals are evaluated by this method whenever it applies.

(d) *Bodies with an axis* such that there is a set of parallel planes which cut from the body plates whose centers of mass lie upon the axis. Taking the axis for x-axis, and the other axes in a plane

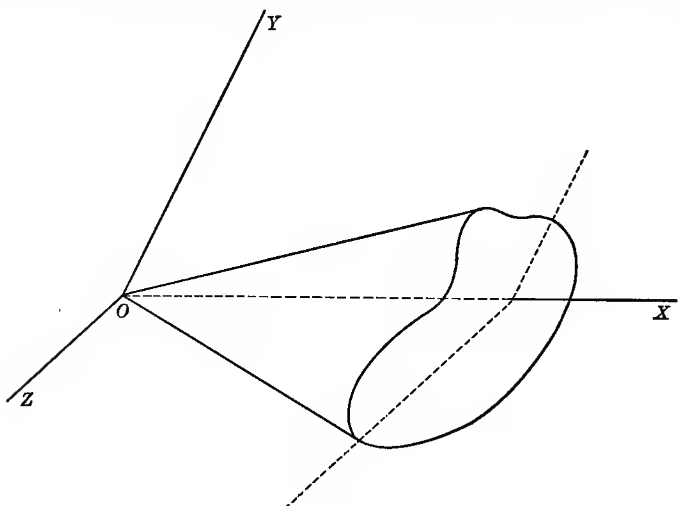


Fig. 13

parallel with the set of planes mentioned (the axes may be oblique; compare the exercise at the end of § 13), these bodies admit of a treatment like wires. Then $\delta(x)$ means the mass of the body between two of the planes of the set, corresponding to x and to $x + \Delta x$, divided by Δx, or, if this ratio varies, its limit. The same formulas again hold.

Example. *Any homogeneous cone, right or oblique, has its center of mass $\frac{3}{4}$ the distance from its vertex to its base.* Planes parallel with the base make

similar sections whose areas vary with the square of the distance from the vertex. We have, therefore, $\delta(x) = kx^2$, whence

$$\bar{x} = \frac{k\int_0^l x^3 dx}{k\int_0^l x^2 dx} = \frac{3}{4}l.$$

(e) *Surfaces of revolution.* Since a surface has no thickness, material surfaces are idealizations, though they are nearly realizable in the case of bodies made of thin metal. We shall suppose the material of constant thickness and density. Then, the x-axis being chosen to coincide with the axis of revolution, and the meridian curve being given by $y = f(x)$, we have $\Delta c = \sqrt{\Delta x^2 + \Delta y^2}$ for a chord of the meridian curve, $2\pi(y + \Delta y/2)\Delta c$ for the surface formed by revolving the chord about the axis (see the section on surfaces of bodies of revolution in the Integral Calculus), so that if d is the surface density, $2\pi(y + \Delta y/2)\sqrt{1 + (\Delta y/\Delta x)^2}\Delta x \cdot d$ is the mass of such a surface. Thence

$$\bar{x} = \frac{\lim \sum x\, 2\pi d \cdot \left(y + \frac{\Delta y}{2}\right)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x}{\lim \sum 2\pi d \cdot \left(y + \frac{\Delta y}{2}\right)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x}$$

$$= \frac{\int xy\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx}{\int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx},$$

and, by symmetry, $\bar{y} = 0$, $\bar{z} = 0$.

Compare this with the derivation of the formula in calculus for the area of a surface of revolution. It need scarcely be pointed out that the distinction between this case and case (c) is that here a *shell* is meant, whereas in case (c) the solid *bounded* by such a shell was intended.

(f) *Curved wires.* The equations of the curve which give the form of the wire should be put into parameter form, say, $x = f(t)$, $y = g(t)$, $z = h(t)$. Then we have for a chord

$$\Delta c = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}\Delta t.$$

Hence the mass of a short piece of the arc is

$$\delta(t)\,\Delta s = \delta(t)\frac{\Delta s}{\Delta c}\Delta c = \delta(t)\frac{\Delta s}{\Delta c}\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2+\left(\frac{\Delta z}{\Delta t}\right)^2}\,\Delta t.$$

We therefore have, remembering that $\lim \Delta s/\Delta c = 1$,

$$\bar{x} = \frac{\lim\sum x\delta(t)\,\Delta s}{\lim\sum \delta(t)\,\Delta s} = \frac{\lim\sum x\delta(t)\left(\frac{\Delta s}{\Delta c}\right)\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2+\left(\frac{\Delta z}{\Delta t}\right)^2}\,\Delta t}{\lim\sum \delta(t)\left(\frac{\Delta s}{\Delta c}\right)\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2+\left(\frac{\Delta z}{\Delta t}\right)^2}\,\Delta t}$$

$$= \frac{\int x\delta(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt}{M}$$

$$= \frac{\int f(t)\,\delta(t)\sqrt{f'^2(t)+g'^2(t)+h'^2(t)}\,dt}{M};$$

and similarly,

$$\bar{y} = \frac{\int g(t)\,\delta(t)\sqrt{f'^2(t)+g'^2(t)+h'^2(t)}\,dt}{M},$$

$$\bar{z} = \frac{\int h(t)\,\delta(t)\sqrt{f'^2(t)+g'^2(t)+h'^2(t)}\,dt}{M},$$

where

$$M = \int \delta(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int \delta(t)\sqrt{f'^2(t)+g'^2(t)+h'^2(t)}\,dt.$$

If possible, the length of the curve should be taken as parameter, for then the radical reduces to

$$\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2} = \frac{ds}{ds} = 1.$$

(g) *Bodies whose bounding surfaces are simpler in some special coördinate system may be treated by the use of such a system.* We illustrate by a plate bounded by a cylindric surface whose generating

curve is given in polar coördinates (see the section on areas bounded by curves given in polar coördinates in the Integral Calculus). The formulas become (cf. case (a))

$$\bar{x} = \frac{\iint \rho \cos\theta\, \delta(\rho, \theta)\, \rho d\rho d\theta}{M},$$

$$\bar{y} = \frac{\iint \rho \sin\theta\, \delta(\rho, \theta)\, \rho d\rho d\theta}{M},$$

$$\bar{z} = 0,$$

where $$M = \iint \delta(\rho, \theta)\, \rho d\rho d\theta.$$

Example. Find the center of mass of a plate of uniform density bounded by the cardioid $\rho = a(1 - \cos\theta)$. We have

$$\bar{x} = \frac{\int_0^{2\pi} \cos\theta \int_0^{a(1-\cos\theta)} \rho^2 d\rho d\theta}{\int_0^{2\pi}\int_0^{a(1-\cos\theta)} \rho d\rho d\theta},$$

for $\delta(\rho, \theta)$, being constant, may be taken from under the integral sign and divided out. These integrals give upon evaluation

$$\bar{x} = \frac{-5a^3\pi}{4} \div \frac{3a^2\pi}{2} = -\frac{5}{6}a.$$

By symmetry, $\bar{y} = 0$.

Notice that in all cases where the density is constant it divides out from the numerators and denominators of the expressions for the coördinates of the center of mass.

(h) *Simple composite bodies* composed of parts whose masses and centers of mass are already known. We imagine the parts replaced by particles of the same masses situated at the respective centers of masses, and apply the formulas of § 12. The idea of *negative* masses is helpful here, a negative mass corresponding to a part *cut out* from a body. Consider, for instance, a body of mass m_1 with center of mass at (x_1, y_1, z_1) from which a part m_2 with center of mass at (x_2, y_2, z_2) is cut out; the resulting body will have a mass $M = m_1 - m_2 = m_1 + (-m_2)$; let the center of mass be at $(\bar{x}, \bar{y}, \bar{z})$.

As the body of mass m_1 may be considered as composed of the other two, we have, by § 12,

$$x_1 = \frac{M\bar{x} + m_2 x_2}{M + m_2},$$

in which, however, $\bar{x}$ is the unknown. Solving, we have

$$\bar{x} = \frac{(M + m_2)x_1 - m_2 x_2}{M} = \frac{m_1 x_1 - m_2 x_2}{m_1 - m_2} = \frac{m_1 x_1 + (-m_2 x_2)}{m_1 + (-m_2)};$$

that is, the same formula as before, except that m_2 is replaced by its negative.

Example. From a homogeneous sphere of radius a is cut out a sphere of radius $b\,(\leqq a/2)$ with center at the mid-point of a radius of the larger sphere. Find the center of mass. Calling the density 1 (cf. the last example), we have a body of mass $m_1 = 4/3\,a^3$ with center of mass at the origin, and a body of mass $-m_2 = -4/3\,b^3$ with center of mass at $(a/2, 0, 0)$. Whence

$$\bar{x} = \frac{\frac{4}{3}\pi a^3 \cdot 0 - \frac{4}{3}\pi b^3 \frac{a}{2}}{\frac{4}{3}\pi a^3 - \frac{4}{3}\pi b^3} = -\frac{2ab^3}{a^3 - b^3}, \quad \bar{y} = 0, \quad \bar{z} = 0.$$

If the student desires to avoid negative masses, he may use the formula $\bar{x} = (m_1 x_1 + m_2 x_2)/M$, where M is the mass of the *whole* body, and $\bar{x}$ the x-coördinate of its center of mass. The resulting equation should then be solved for x_1. Note that $m_1 = M - m_2$.

VI. PROBLEMS ON CENTERS OF MASS OF CONTINUOUS BODIES (*Continued*)

7. Find the center of mass of a straight wire of length l whose density varies

(a) as the distance from one end;
(b) as the square of the distance from one end;
(c) as the nth power of the distance from one end;
(d) as $(l^2 - x^2)$;
(e) as $\sin(\pi x/l)$;
(f) as $\sin(\pi x/2l)$.

8. Find the center of mass of a plate in the shape of a quarter ellipse bounded by semimajor and semiminor axes.

9. Find the center of mass of a plate bounded by the parabola $y^2 = 4ax$ and the chord $x = h$. Particular case where the chord is the latus rectum $(h = a)$.

10. Find the center of mass of three faces of a cube regarded as plates, (a) when the faces all meet in a point; (b) when they do not.

11. Find the center of mass of the cone formed by revolving $y = mx$ about the x-axis, between the vertex and the base plane $x = h$.

12. Find the center of mass of the body between the paraboloid formed by revolving $y^2 = 4ax$ about its axis and the plane $x = h$.

13. Find the same when the density varies with $1/x$.

14. Find the center of mass of the homogeneous hemisphere by method of case (c).

15. Find the center of mass of half of the ellipsoid of revolution obtained by revolving $x^2/a^2 + y^2/b^2 = 1$ about the x-axis, the bounding plane being the plane $x = 0$.

16. Find the center of mass of the plate bounded by the x-axis and an arch of the sine curve $y = \sin x$.

17. Find the center of mass of a regular tetrahedron composed of four equilateral triangular plates.

Hint. Use oblique axes and regard the plates as concentrated at their centers of mass.

18. Find the center of mass of a solid homogeneous tetrahedron.

Hint. Take origin at a vertex and x-axis through the point of intersection of the medians, i.e. at the centroid of the opposite face. Use method of case (d).

19. Find the center of mass of a hemispherical bowl of radius r.

20. Find the center of mass of a conical surface formed by revolving $y = mx$ about the x-axis, between the planes $x = a$ and $x = b$.

21. Find the center of mass of the wire bent into the form of a circular arc of radius r and angle $2a$. Check by putting $a = \pi$.

Hint. Polar coördinates advised.

22. Find the center of mass of a wire bent into the form of a cardioid $\rho = a(1 - \cos\theta)$.

Hint. Use half angles.

23. Find the center of mass of a spring $x = a\cos t$, $y = a\sin t$, $z = amt$, the spring to be ended by the planes $z = 0$ and $z = h$. Show that the centroid lies on the axis only when the spring makes an integral number of complete turns.

24. Find the center of mass of the wire $\rho = e^{a\theta}$ from $\theta = 0$ to $\theta = 2\pi$.

25. From a right circular cylinder of height h and base of radius r is cut a cone of the same base and altitude. Find the center of mass of the remaining body.

26. Find the center of mass of a hemispherical shell bounded by a plane and two concentric spheres of radii a and b about one of its points.

27. Show that the center of mass of the body between two right circular cones with same vertex, axis, and base plane, but different angles, coincides with the center of mass either cone would have if solid.

28. Find the center of mass of a plate in the form of a segment of a circle of radius r, subtending an angle $2a$ at the center.

29. By adding the proper triangle to the segment, get the center of mass of the sector and thus check the results of problems 6 and 28.

30. Find the center of mass of the body formed by revolving the oval of $y^2 = x(x-a)(b-x)$, $(0<a<b)$, about the x-axis.

31. From a square plate an equilateral triangle with one side of the square for its base is removed. Find the center of mass of the remaining plate.

32. From an elliptic plate $x^2/a^2 + y^2/b^2 = 1$ a circular plate of radius r with center at the mid-point of the semi-major axis is removed. Find the center of mass of the remaining plate.

33. To the elliptic plate of the above problem are attached two circular plates of half the thickness of the elliptic plate on each side, so as to have the effect of doubling its thickness where previously the material was removed. Find the center of mass.

34. Find the center of mass of a wire in the shape of an arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$.

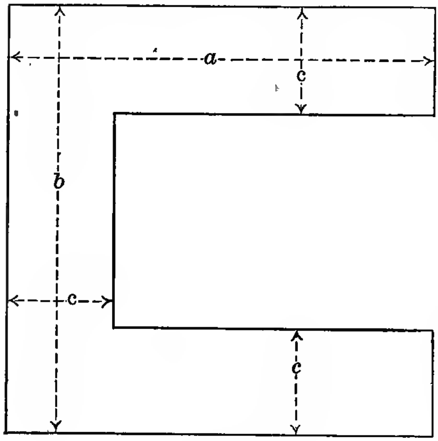


FIG. 14

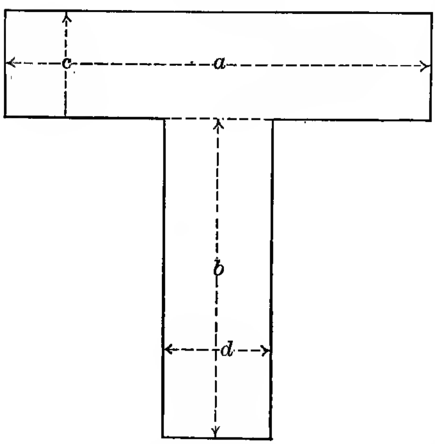


FIG. 15

35. Find the center of mass of a triangular plate.

Hint. Use oblique axes, one axis in the base of the triangle and the other the median upon it. If rectangular axes are used, take the vertices at $(a, 0)$, $(b, 0)$, $(0, c)$.

36. Find the center of mass of the plate bounded by the cardioid $\rho = a(1-\cos\theta)$.

37. Find the center of mass of the plate bounded by half the lemniscate $\rho^2 = a^2 \cos 2\theta$.

38. Find the center of mass of the plate bounded by one loop of $\rho = a \sin 2\theta$.

39. Show that the center of mass of the surface of a zone of a sphere is halfway between the bounding planes.

40. Find the center of mass of an arc of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (or $x = a\cos^3 t$, $y = a\sin^3 t$) between two successive cusps.

41. Find the center of mass of half a right circular cone, the dividing plane passing through the axis.

42. Find the center of mass of the plates indicated in Figs. 14–17.

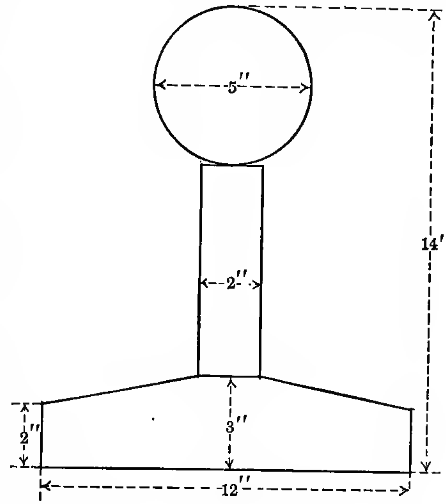


FIG. 16

43. Find the center of mass of a wire in the form of a catenary $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$ from $x = 0$ to $x = a$.

44. Find the center of mass of a wire given by the equations $y = \frac{2}{3}\sqrt{x^3/a}$, $z = x^2/4a$ (or $x = at^2$, $y = 2at^3/3$, $z = at^4/4$), from the origin to the point $(a, 2a/3, a/4)$.

45. Find the center of mass of a plate in the form of a sector of the logarithmic spiral $\rho = be^{m\theta}$ bounded by the radii $\theta = 0$ and $\theta = a$.

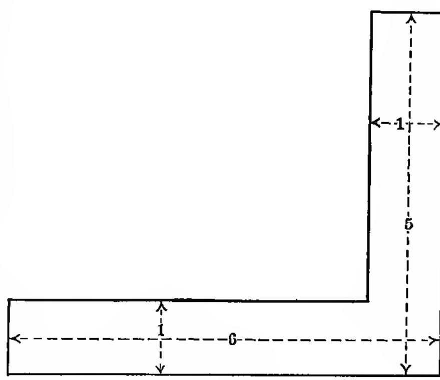


FIG. 17

46. Find the center of mass of the search-light reflector obtained by revolving the parabola $y^2 = 4ax$ about the x-axis, and bounded by the plane $x = a$.

47. Find the center of mass of a spherical wedge bounded by two planes meeting at an angle $2a$ and a sphere of radius r with center on their line of intersection.

Hint. Split up into plates normal to edge of the wedge and use problem 6.

48. Prove the *first theorem of Pappus:* The area of the surface generated by revolving a plane curve $y = f(x)$ between the ordinates $x = a$ and $x = b$ about the x-axis is equal to the product of the length of the arc and the length of the path described by the rotation about the axis of the arc.

49. Prove the *second theorem of Pappus:* The volume of the body within the surface of the above problem is the area under the curve times the length of the path described by the rotation about the axis of the center of mass of the area.

Hint. Use double integrals.

50. Find by the theorems of Pappus the surface and volume of the torus, formed by rotating about an axis a circle of radius a whose center is a distance b from the axis $(b > a)$.

51. Find the center of mass of the part included in a sphere of radius r and a right circular cone whose elements make an angle a with its axis, and whose vertex is at the center of the sphere.

52. Find the center of mass of the body bounded by a sphere of radius r and a right circular cone with vertex at the center of the sphere, and whose elements make an angle a with its axis. Verify by putting $a = \pi/2$ and comparing with the result of problem 14.

53. Find the center of mass of a spherical cap bounded by a sphere of radius r and a plane distant $r \cos a$ from the center. Verify as in the preceding problem.

54. Find the center of mass of the part remaining of a hemisphere of radius r after removing the part contained within a circular cylinder of radius a, and whose axis coincides with the axis of the circle forming the edge of the hemisphere.

55. Find the center of mass of the plate bounded by the x-axis and an arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

56. Treat problems 4 and 5 by the method of case (d).

57. Find the center of mass of a parabolic wire $y^2 = 4\,ax$ bounded by the latus rectum. The answer also holds for the right cylindric surface of which the wire is generatrix, and which is bounded by two planes perpendicular to the elements.

58. Find the center of mass of the part remaining of a sphere after removing the part contained within a cone with axis a diameter and vertex on the surface, the elements of the cone making an angle a with its axis.

59. From the problems on areas and volumes of surfaces of revolution in your Calculus text, find by means of the theorems of Pappus the center of mass of the corresponding curves and areas (wires and plates).

ANALYSIS OF CHAPTER II

1. Subject-matter of statics.

2. Equilibrium defined.

3. The two conditions for equilibrium of concurrent forces.

4. Analytic statement of the condition for equilibrium of concurrent forces.

5. Assumptions regarding frictional forces.

6. Definition of the moment of a force about an axis; its value for different positions of the axis.

7. Definition of couple. Constancy of the moment of a couple when the axis is shifted parallel to itself.

8. The shifting of the point of application of a force by the introduction of a couple.

9. Conditions for the equilibrium of nonconcurrent forces.

10. Analytic statement of the first conditions for equilibrium.

11. Analytic statement of the second conditions for equilibrium.

12. Definition of center of mass and its property with respect to a system of parallel forces proportional to the masses of the parts.

13. Formulas for the centers of mass of a system of particles and of continuous bodies.

CHAPTER III

DYNAMICS OF A PARTICLE

15. Dynamics. Dynamics treats of the effect of forces in producing motion. We shall here limit ourselves principally to the motion of particles in a plane, for we thereby obtain methods applicable to the majority of interesting problems and avoid unnecessary complications of treatment. Furthermore, we start with the simplest kind of motion in a plane, namely, motion in a straight line.

16. Rectilinear motion; concepts involved. Take as origin a convenient point O on the line, and call the distance OP of the particle at P from O, s; then the fundamental notions involved in the motion are the distance s, the time t, the velocity v, and the acceleration a.* To these should be added the mass m of the body and the force f acting upon it. As, however, the mass is constant and $f = ma$, we have really to deal essentially only with the four quantities above mentioned, and the problems of straight-line motion are usually concerned with *a relation between two or more of these.* We proceed to consider the more important cases that may arise, and to indicate how they should be treated.

(a) *A relation given between distance and time,* say $s = f(t)$. This we shall call a *complete description* of the motion, because the

* *Velocity.* If the motion is *uniform,* a distance traversed, $s_2 - s_1$, is proportional to the time consumed, $t_2 - t_1$. The velocity is then defined as the constant ratio $v = (s_2 - s_1)/(t_2 - t_1) = \Delta s/\Delta t$. If the motion is not uniform, this ratio depends upon both t_1 and t_2, and is called the *average velocity during the interval t_1 to t_2.* By the *velocity at the time t* we mean the limit of the average velocity over an interval starting with t as the interval is shortened indefinitely, i.e. $v = \lim \Delta s/\Delta t = ds/dt$.

Acceleration. Force tends to change "motion," i.e. to change velocity. The rate of change is called acceleration. If the motion is "uniformly accelerated," $(v_2 - v_1)/(t_2 - t_1) = \Delta v/\Delta t$ is constant and defines the acceleration. If this ratio is not constant, it is called the *average acceleration during the interval t_1 to t_2.* By the *acceleration at the time t* we mean the limit of the average acceleration over an interval starting with t as the interval is shortened indefinitely, i.e. $a = \lim (\Delta v/\Delta t) = dv/dt = ds/dt$.

position of the body is known at every time, and the questions arising, including those concerning velocity and acceleration, are answered by means of differentiations. All other cases involve *integrations.* Moreover, in this case only does the given relation suffice to determine *uniquely* the motion, for there are no constants of integration here. *The problem in any other case may be considered as essentially solved by reduction to this one; that is, by deriving a relation*

$$s = f(t).$$

ILLUSTRATIVE EXERCISE. Consider the motion $s = t^3 - 3t$. Find v and a. When does the body come to rest, and at what points? How does it move before, between, and after these times of rest? When is the force to the right and when to the left? Discuss similarly the motion $s = 3t^5 - 20t^3 + 10$.

(b) *A relation given between acceleration (or force) and time,* say, $a = f(t)$. To reduce this back to the complete description of case (a) requires two integrations. As $a = dv/dt$, we have

$$\frac{dv}{dt} = f(t), \quad \text{whence} \quad v = \int f(t)\,dt + c_1,$$

and as $v = ds/dt$, we have

$$\frac{ds}{dt} = \int f(t)\,dt + c_1, \quad \text{whence} \quad s = \iint f(t)\,dt\,dt + c_1 t + c_2.$$

The constants cannot be determined without auxiliary conditions, which usually accompany each problem. They will always be given in the exercises below.

EXAMPLE. A body is repelled from a fixed point by a force proportional to the time t. The body starts from rest at the origin when $t = 0$. Find the complete description. The conditions of the problem are $f = kt$, whence $ma = kt$; and $v = 0$, $s = 0$, when $t = 0$. The first gives $dv/dt = (k/m)t$, whence $v = (k/2m)t^2 + c_1$. But as $v = 0$ when $t = 0$, we have $0 = 0 + c_1$. Whence $c_1 = 0$ and $v = (k/2m)t^2$. But $v = ds/dt$. Whence $ds/dt = (k/2m)t^2$ and $s = (k/6m)t^3 + c_2$. But as $s = 0$ when $t = 0$, we have $c_2 = 0$, whence $s = (k/6m)t^3$, the required relation.

(c) *A relation given between acceleration (or force) and distance,* say, $a = f(s)$. This also requires two integrations, which may be carried out as follows: Write the relation $dv/dt = f(s)$, and multiply

by $2v$ on the left and by its equivalent $2(ds/dt)$ on the right, obtaining $2v(dv/dt) = 2f(s)(ds/dt)$. Whence, integrating with regard to t, we have

$$v^2 = \int 2f(s)\,ds + c_1.$$

To carry out the second integration, solve for v, obtaining

$$v = \frac{ds}{dt} = \sqrt{2\int f(s)\,ds + c_1},$$

whence $dt = \dfrac{ds}{\sqrt{2\int f(s)\,ds + c_1}}$ and $t = \displaystyle\int \frac{ds}{\sqrt{2\int f(s)\,ds + c_1}} + c_2.$

If, after determining c_1 and c_2 by the auxiliary conditions given, and carrying out the integrations, the resulting equation is easily solvable for s, we have our *complete description* of case (a). Otherwise we have a, v, and t all expressed as functions of s, which may be considered as giving a parameter representation of the motion.

EXAMPLE. A body falls toward the origin under a force inversely proportional to the square of the distance away. If v approaches 0 as s approaches ∞, and if $s = 0$ for $t = 0$, determine the motion. Considering the body to move on the positive part of the line, the force will be negative, as it is toward the origin. A proportionality factor giving simple results is $-km/2$. Then, as $f = -km/2s^2$, and $ma = f$, we have $a = dv/dt = -k^2/2s^2$. Then, as above, $2v(dv/dt) = -(k^2/s^2)(ds/dt)$, or $v^2 = k^2/s + c_1$. But $v \doteq 0$ as $s \doteq \infty$, hence $0 = 0 + c_1$ and $v^2 = k^2/s$, whence $v = \pm k/\sqrt{s}$. The sign to choose is the negative one, as the body is falling toward the origin. Hence $v = ds/dt = -k/\sqrt{s}$ and $\sqrt{s}\,ds = -k\,dt$. Hence $\frac{2}{3}s^{\frac{3}{2}} = -kt + c_2$, where $c_2 = 0$ by the auxiliary conditions. Thus the motion is given by

$$s = \left(-\frac{3kt}{2}\right)^{\frac{2}{3}}.$$

Among some interesting questions that may be attached to the above problem are: (1) Has the body fallen from an infinite distance in finite time? (2) With what velocity does it reach the origin? (3) Does the solution given hold for positive or negative t? (4) Has the solution a meaning when t has the other sign, and if so, what is it?

(d) *A relation given between acceleration (or force) and velocity*, say, $a = f(v)$. Two integrations are necessary. First

$$\frac{dv}{dt} = f(v), \quad \text{whence} \quad \int \frac{dv}{f(v)} = t + c_1.$$

This relation must then be solved for v, giving, say, $v = F(t + c_1)$. Thence

$$s = \int F(t + c_1)\,dt + c_2.$$

A relation between the velocity and the distance also may be found by one integration. Thus, as

$$a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\frac{dv}{ds},$$

we have $$v\frac{dv}{ds} = f(v), \quad \text{whence} \quad \int \frac{vdv}{f(v)} = s + c.$$

The formula $a = v\,(dv/ds)$ is a very useful one.

(e) *A relation given connecting acceleration (or force), velocity, and distance.* In the more interesting problems this relation is linear in a, v, and s, and has constant coefficients, say, $a + bv + cs = 0$ or $d^2s/dt^2 + b\,(ds/dt) + cs = 0$, for the integration of which the student is referred to text-books on the calculus or on differential equations.

Graphic representations of a motion. (a) On a straight line. Plot the points corresponding to a succession of values of t differing by equal increments. The direction of motion may be indicated by arrows parallel to the line. The closeness together of the points gives an indication of the speed. If the motion returns upon itself, draw a number of parallel lines and on each one plot the motion between two of its turning points. If in doing this we imagine the points actually marked at equal intervals of time, the point of the pencil has a motion which is a picture of the required motion.

(b) Using the plane. Draw a t-axis perpendicular to the s-axis and plot $s = f(t)$. If now a slit be cut in a piece of paper, and the plot be put behind the slit, so that the slit is parallel with the s-axis, and if the plot be drawn with uniform speed downward in the direction of the negative t-axis, the point of the plot which shows through the slit will appear with the required motion. The maxima and minima of $s = f(t)$ will be the turning points, and the speed is given by the slope of the curve with respect to the t-axis. In discussing the motion the student should use one of these methods, and it is highly valuable for him to become acquainted with both (see Fig. 18).

VII. PROBLEMS ON RECTILINEAR MOTION

In the first few of the following problems the "complete description" is given and the motion should be discussed. A discussion should include such points as the following:

(a) Times and points where the motion stops, i.e. where the velocity vanishes.

(b) The points reached by the motion; e.g. in $s = t^2$ only the points to the right of the origin, with the origin.

(c) Direction of the motion between stops (judged by the sign of v).

(d) The number of times each part of the line is traversed, e.g. in the above example the positive half of the line is traversed twice, the negative half not at all.

(e) Direction of the acceleration (i.e. of the force).

(f) Tendency of the motion for large negative and for large positive values of t.

It may be that not all these characteristics will be of interest in a given motion, whereas special motions will have other characteristics of special interest, — which should be pointed out. The student should in advance get clearly in mind the meanings of sign of the velocity and acceleration. This may be done by answering the question, What is characteristic in the following four types of motion: (1) $v>0$, $a>0$; (2) $v>0$, $a<0$; (3) $v<0$, $a>0$; (4) $v<0$, $a<0$?

1. $s = t^3 - 6t^2 + 10$.

Solution. $v = 3t^2 - 12t = 3t(t-4)$. $a = 6t - 12 = 6(t-2)$. (a) The motion ceases for $t = 0$ and $t = 4$, that is, at the points $s = 10$ and $s = -22$.

(b) All points are reached, because a cubic equation $t^3 - 6t^2 + 10 = s$ will have a real root, no matter what the value of s.

(c) The direction of motion depends upon the sign of v, which depends upon the signs of its factors. For $t<0$, $v>0$, and the motion is forward to $s = 10$. For $0<t<4$, $v<0$, and the motion is backward to -22. For $4<t$, $v>0$, and the motion is forward always thereafter.

(d) The points between -22 and $+10$ are traversed three times, other points once.

(e) The acceleration is backward till $t = 2$, at the point $s = -6$, then forward.

(f) The farther back in time we go, the farther to the left was the point (because for large negative t, s is large and negative), and there is no point to the left of which the moving point has not been. The velocity is positive for negative t, but decreasing as time progresses; that is, the motion is a slower and slower forward motion. For larger and larger positive t we find s increasing without limit and v also; that is, the point moves forward beyond any point whatever with an always increasing velocity.

Graphic representations of the motion are shown in Fig. 18.

2. $s = -16t^2 - 32t - 10$.
3. $s = bt + s_0$.
4. $s = 2t^3 - 3t^2 - 12t + 12$.
5. $s = 1/t$.
6. $s = 1/(t^2 + 1)$.
7. $s = 1/(t^2 - 1)$.
8. $s = b \sin t$.
9. $s = b \sin(kt + e)$.
10. $s = e^t$.
11. $s = e^{-t}$.
12. $s = \log t$.
13. $s = e^{-t} \cdot \sin t$.
14. $s = e^{-1/t^2}$.
15. $s = b(t-1)^3$.
16. $s = b \cdot \tan t$.
17. $s = \frac{b}{2}(e^{kt} + e^{-kt})$.
18. $s = \frac{b}{2}(e^{kt} - e^{-kt})$.
19. $s = b(e^{kt} - e^{-kt})/(e^{kt} + e^{-kt})$.
20. $t = s^3 - 6s - 1$.
21. $t = s + \sin s$.

Can a point actually have the motion given by problems 20 and 21?

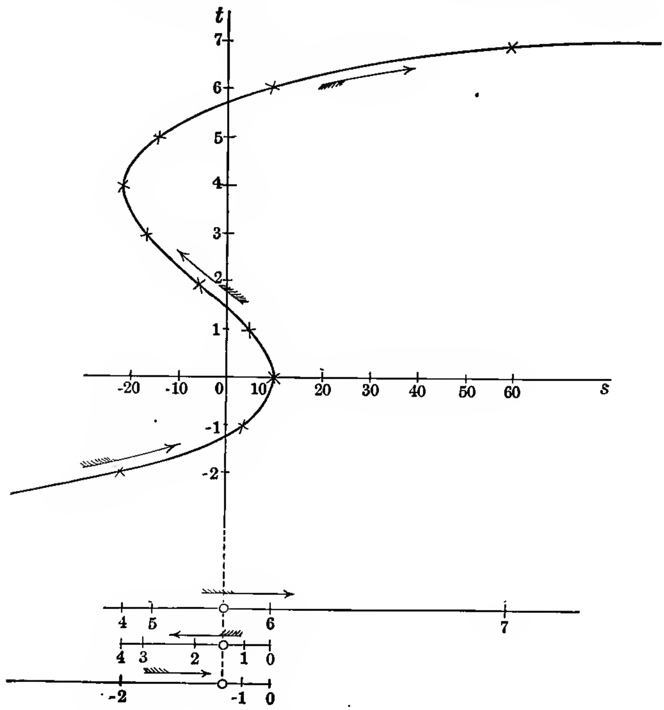


FIG. 18

In the next problems find the "complete description," determining the constants of integration by the auxiliary conditions given. If the motion is not recognized as one of the above types, it should also be discussed in the same way. Tell in all cases the force involved.

22. $v = kt$; $s = 1$ when $t = 0$.

23. $v = ks$; $s = s_0$ when $t = t_0$.

24. $a = kv$; $v = 2$, $s = 1$ when $t = 0$.

25. $a = kt$; $s = 0$ when $t = 0$, $s = 0$ when $t = 1$.

26. $a = k^2s$; $v = 0$, $s = a$ when $t = 0$.

27. $a = k^2s$; $v = ka$, $s = 0$ when $t = 0$.

28. $a = -k^2s$; $v = v_0$, $s = 0$ when $t = t_0$.

29. $a = -k^2/s^2$; v approaches 0 as s approaches ∞, $s = s_0$ when $t = 0$. Show that in this problem the body can only move on the positive side of the origin.

30. $a = -2bv + cs$; obtain the general solution and discuss for $b^2 - c \gtreqless 0$.

17. Some special rectilinear motions. (a) *The inclined plane.* Let i denote the angle of inclination of the plane to the horizon. Then, resolving the force of gravity into two components, perpendicular and parallel to the plane respectively, the parallel component, $mg \cos i$, is alone effective in producing motion. The further development of this topic is left to the student in the exercises and problems to follow.

Ex. 1. Determine the motion of a body sliding from rest down an inclined plane of inclination i.

Ex. 2. Show that bodies starting simultaneously to slide down various chords of a vertical circle from the highest point of the circle will all reach the circumference at the same time.

(b) *Simple harmonic motion.* Let a body be attracted toward a fixed center with a force varying with its distance away from the center.* Let the proportionality factor be mk^2; then, as the sign of the acceleration is opposite to that of s, we have $a = -k^2s$, or $d^2s/dt^2 + k^2s = 0$. If this be integrated as a linear equation with constant coefficients (see also the exercise below), we have for the solution $s = c_1 \cos kt + c_2 \sin kt$. It is shown in Analytic Geometry that two numbers c_1 and c_2 are always proportional to the sine and cosine of some angle e, say, $c_1 = A \sin e$, $c_2 = A \cos e$. Then s becomes $s = A \sin(kt + e)$. The student should verify by direct

* Examples of forces of this sort are elastic forces, for small displacements, due to springs and elastic bands; gravity in mines and beneath the surface of the earth; and, approximately, the forces involved in pendulum motion and the motion of magnetic needles.

substitution that this satisfies the differential equation. This motion the student has studied in problem 9 of the previous paragraph. The motion repeats itself at intervals such that

$$kt_{n+1} + e = kt_n + e + 2\pi, \quad \text{or} \quad t_{n+1} - t_n = T = \frac{2\pi}{k}.$$

This quantity T is called the *period* of the simple harmonic motion.

Ex. 3. Integrate the above differential equation by the method of case (c) in § 16 and reconcile the results.

Ex. 4. Discuss *dampened harmonic motion* in which there is a resistance due to air or friction proportional to the velocity, so that $a = -k^2s - bv$. Compare with problem 13 of § 16, which is a special case.

(c) *Fall of a body from a great height.* The force of gravity acting upon a body is sensibly constant if the distance through which the body moves is small. Careful measurements, however, do reveal a variation, and, according to Newton's law of universal gravitation, the force upon a body outside the earth's surface varies with the inverse square of the distance from the earth's center.

Let us take for origin the point from which the body falls, say, at a height h from the earth's center, and let the direction toward the center be taken as positive. Then $h - s$ is the distance of the body from the center, and $f = ma = mc/(h-s)^2$. If R denote the radius of the earth, we have, since at the surface $a = g$, $g = c/R^2$, or $c = gR^2$. Thus our equation becomes

$$a = \frac{gR^2}{(h-s)^2}, \quad \text{or} \quad \frac{d^2s}{dt^2} = \frac{gR^2}{(h-s)^2},$$

which comes under case (c) of the last paragraph. We have, then,

$$v\frac{dv}{dt} = gR^2 \frac{1}{(h-s)^2}\frac{ds}{dt}, \quad \text{whence} \quad \frac{v^2}{2} = +\frac{gR^2}{h-s} + c_1.$$

As the body falls from rest, $v = 0$ when $s = 0$, so that

$$0 = g\frac{R^2}{h} + c_1,$$

and therefore $v^2 = 2\,gR^2\left(\frac{1}{h-s} - \frac{1}{h}\right) = 2\,gR^2\left(\frac{s}{h(h-s)}\right).$

Whence $$v = \frac{ds}{dt} = R\sqrt{\frac{2g}{h}}\cdot\sqrt{\frac{s}{h-s}}.$$

This admits of further integration by the ordinary methods, but the integral is more useful when obtained in the form of a series. We have

$$dt = \frac{1}{R}\sqrt{\frac{h}{2g}}\left(\frac{h-s}{s}\right)^{\frac{1}{2}}ds.$$

For the motion which interests us $0 < s < h$, so that we develop the radical as a series in s/h.

$$\begin{aligned}\left(\frac{h-s}{s}\right)^{\frac{1}{2}} &= \left(\frac{h}{s}\right)^{\frac{1}{2}}\left(1-\frac{s}{h}\right)^{\frac{1}{2}}\\ &= \left(\frac{h}{s}\right)^{\frac{1}{2}}\left[1-\frac{1}{2}\left(\frac{s}{h}\right)-\frac{\frac{1}{2}\cdot\frac{1}{2}}{1\cdot 2}\left(\frac{s}{h}\right)^2-\frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 2\cdot 3}\left(\frac{s}{h}\right)^3\cdots\right]\\ &= \left(\frac{h}{s}\right)^{\frac{1}{2}}-\frac{1}{2}\left(\frac{s}{h}\right)^{\frac{1}{2}}-\frac{1}{2}\cdot\frac{1}{4}\left(\frac{s}{h}\right)^{\frac{3}{2}}-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\left(\frac{s}{h}\right)^{\frac{5}{2}}-\cdots.\end{aligned}$$

Using this series in the differential equation and integrating, we have

$$t + c_2 = \frac{1}{R}\sqrt{\frac{h}{2g}}\left(2\,(hs)^{\frac{1}{2}}-\frac{2}{3}\frac{s^{\frac{3}{2}}}{h^{\frac{1}{2}}}-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{2}{5}\frac{s^{\frac{5}{2}}}{h^{\frac{3}{2}}}-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\cdot\frac{2}{7}\frac{s^{\frac{7}{2}}}{h^{\frac{5}{2}}}-\cdots\right),$$

or $$t = \frac{h}{R\sqrt{2g}}s^{\frac{1}{2}}\left[2-\frac{s}{h}-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{2}{5}\left(\frac{s}{h}\right)^2-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\cdot\frac{2}{7}\left(\frac{s}{h}\right)^3-\cdots\right],$$

where we have put $c_2 = 0$ on the supposition that $s = 0$ when $t = 0$; that is, that we count the time from the instant the body starts to fall. If s is small compared with h, this series converges and a few terms give a good approximation to the true motion.

Ex. 5. Write out the nth term of the above series.

Ex. 6. Taking simply the first term of the series above and putting $h = R$, show that we have the ordinary law for falling bodies at the earth's surface.

Ex. 7. The terms of the series after the first are all negative, so that the actual motion differs from that considered in the preceding exercise in a way easy to state. Make the statement.

(d) *Body falling through a resisting medium.* Raindrops and meteorites are examples of this case; similar forces act upon ships

and trains moving with small velocities under propulsion of their own engines. The resistance we shall assume to be proportional to the square of the velocity, and take our s-axis downward. We have, then,

$$a = g - \frac{g}{k^2} v^2,^*$$

where g is the acceleration due to gravity and g/k^2 a convenient proportionality factor. The differential equation is, writing $a = dv/dt$,

$$\frac{dv}{v^2 - k^2} = -\frac{g}{k^2} dt,$$

whence $$\frac{1}{2k} \log \frac{v-k}{v+k} = -\frac{g}{k^2} t + \text{const.}, \quad \text{or} \quad \frac{v-k}{v+k} = c_1 e^{-\frac{2gt}{k}}.$$

This shows clearly a tendency of the motion, namely, as time goes on $e^{-(2g/k)t}$ grows rapidly small, and hence v approaches k, that is, the velocity tends to become constant. The constant k, introduced above, thus receives as its interpretation this limiting velocity. If the body fall from rest, we have $v = 0$ for $t = 0$, so that $-1 = c_1$, and using this value and solving for v, we have

$$v = \frac{ds}{dt} = k \frac{e^{\frac{2gt}{k}} - 1}{e^{\frac{2gt}{k}} + 1} = k \frac{e^{\frac{gt}{k}} - e^{-\frac{gt}{k}}}{e^{\frac{gt}{k}} + e^{-\frac{gt}{k}}},$$

whence $$s - s_0 = \frac{k^2}{g} \log \left(\frac{e^{\frac{g}{k}t} + e^{-\frac{g}{k}t}}{2} \right).$$

As noted before, $e^{-(g/k)t}$ tends toward zero as t increases, and we obtain an approximation by dropping it. The result of this approximation is the uniform motion $s = kt + const.$

Ex. 8. Show that if the body have a downward initial velocity of v_0, the integral will be $s - s_0 = k \log [v_0 \sinh (g/k) t + k \cosh (g/k) t]$ where

$$\sinh u = \frac{e^u - e^{-u}}{2}, \quad \text{and} \quad \cosh u = \frac{e^u + e^{-u}}{2}.$$

Ex. 9. Study the motion of a body on which no constant force, but the resistance alone, acts, there being an initial velocity v_0.

* This equation holds only during downward motion, for if the velocity were upward, i.e. negative, the resistance is added to the acceleration due to gravity. For upward motion the law is $a = g + gv^2/k^2$. The student should bear in mind that any simple law of resistance like the above is merely an approximation. See Osgood, *Differential and Integral Calculus*, p. 216.

VIII. PROBLEMS ON BODIES MOVING IN STRAIGHT LINES UNDER THE ACTION OF GIVEN FORCES*

1. A body falls from a height 100 m. After falling for 2 sec. a second body is projected vertically upward from the earth toward the first with a velocity 40 m. per sec. Find the time and height at which they meet.

Solution. Let us count the time from the instant the first body begins to fall, and the distance vertically upward from the earth. If we use subscripts to distinguish the two bodies, we then have the conditions $a_1 = -g$, with $v_1 = 0$, $s_1 = 10{,}000$ when $t = 0$ for the first body, the units being centimeters and seconds; and $a_2 = -g$, with $v_2 = 4000$ and $s_2 = 0$ when $t = 2$. These give $s_1 = -gt^2/2 + 10{,}000$, $s_2 = -g(t-2)^2/2 + 4000(t-2)$. The bodies meet when $s_1 = s_2$; equating the two expressions, we find the time of meeting to be $t = 3.35$ sec. nearly, and this gives $s_1 = s_2 = 4500$ cm. or 45 m. about.

Remarks. (a) It is of highest importance that we be *consistent* in our use of units. In the C.G.S. system everything should be reduced to centimeters, grams, and seconds. The value of g is approximately 981 cm. per sec. per sec.

(b) Our problem is usually in these exercises to determine the force acting on a body, to equate it to the mass times the acceleration, and to determine the initial conditions. Thence we determine the complete description of the motion and by means of it answer the questions proposed.

2. A balloon is ascending with a velocity 20 mi. per hr. when a stone is dropped from it. The stone reaches the ground in 6 sec. Find the height of the balloon when the stone was dropped.

3. How high will a stone rise if thrown upward with a velocity 80 m. per sec.?

4. Show that a body thrown upward has the same speed in ascending and descending past a given point.

5. A body thrown into the air with a velocity v_0 attains a certain height before falling. How much must the velocity be increased in order to double this maximum height?

6. A weight 12 lb. on an inclined plane of inclination $i = \arctan 1/2$ is connected by a string to a weight 8 lb. hanging over the upper edge of the plane and starts from rest. Find the distance described in 5 sec.

7. A body weighing 9 kgm., on a smooth table, 3 m. from its edge is connected by a string to a body weighing 1 kgm. and hanging over the edge. Find (a) their common acceleration, (b) the time when the first mass leaves the table, (c) its velocity upon leaving.

8. Two particles of mass m_1 and m_2 are connected by an inextensible string which passes over a frictionless pulley. If $m_2 > m_1$, show that the

* Except when otherwise stated, the force of gravity is to be considered constant and resistances are to be neglected.

tension T on the string is $2m_1m_2/(m_1+m_2)$ and determine the motion of the system. (Note that if s_1 and s_2 are the distances of the particles below the pulley, $s_1+s_2=$ constant and hence $v_1+v_2=a_1+a_2=0$.)

9. Show that during the motion in the above problem the pressure on the axle is less than the sum of the weights of the two bodies. (Be careful to express all forces in the same units. The theorem may be proven by the fact that a perfect square, $(m_2-m_1)^2$, is positive.)

10. Over a pulley passes a string to one end of which is attached a weight 10 lb., and to the other a weight 8 lb. with a rider 4 lb. After being in motion 5 sec. the rider is removed without checking the velocity. How much farther will the system move?

11. With what velocity must a particle be projected downward in order to overtake in 10 sec. a body that has already fallen 100 ft. from rest at the same point?

12. Given a point and a vertical line a distance d from it, find the inclination of the straight line which would guide a particle acted upon by gravity only, from the point to the line in the briefest time.

13. Consider the same problem when the line is not vertical.

14. What is the weight by a spring balance of a man of 160 lb. descending in an elevator with an acceleration 2 ft. per sec. per sec.?

15. A high jumper in jumping raises his center of mass 3 ft. How high could he jump on the surface of the moon, where he weighs one sixth as much? How long is he off the ground in both cases? (Assume he leaves the ground with the same velocity in both cases.)

16. If a motor car with speed 40 mi. per hr. can be stopped by its brakes in a hundred yards, find the inclination of a hill on which the brakes would just hold it.

17. A train with velocity 30 mi. per hr. runs with steam shut off against a resistance of 10 lb. per T. How far will it go?

18. A train running at 15 mi. per hr. with steam shut off strikes an up grade of 1 in 300. The resistance due to friction and air averages 3 lb. per T. Find how far the train will run before stopping.

19. A train of weight 200 T. descends a grade of inclination $i=\arcsin\frac{1}{120}$. If its velocity is initially 40 mi. per hr., what frictional resistance will stop it in half a mile?

20. The attraction of the earth for a particle of mass m beneath the surface of the earth varies as the distance from the earth's center, and it is mg at the surface. Find how soon the body would reach the center if dropped into a hole through the center; also its maximum speed, and at what point this speed is attained.

21. An elastic band is stretched between two points on a smooth table. A weight fixed at the mid-point of the band moves in the line of the band,

the elastic force being proportional to the displacement. Discuss the motion from its equations.

22. To one end of an elastic cord of natural length 2 ft. is attached a weight 2 lb., which when hanging in equilibrium extends the cord to a length $2\frac{1}{2}$ ft. Assuming Hooke's law, that the elastic force exerted by the cord is proportional to its increase in length, determine how far the weight would fall if allowed to drop from the point at which the suspending cord has its natural length 2 ft.

23. A cylinder of radius 5 cm. is weighted at one end so as to float vertically in water. If forced downward into the water a distance 5 cm. below its position of equilibrium and then released, it is found to oscillate vertically with a period 2 sec. Determine the cylinder's weight.

24. Find the velocity with which a body reaches the earth's surface in falling from a height equal to the earth's radius, resistance of the atmosphere being neglected. Find the time occupied by the fall. (Use about five terms of the series.)

25. As we increase the distance through which a body falls to the earth, show that the velocity with which it reaches the surface approaches an upper limit $\sqrt{2\,gR}$. If a body were shot upward with this velocity, what would happen?

26. A ship steaming at its maximum rate of 12 mi. per hr. is stopped by reversed engines and resistance in 6 sec. What is the value of k, and what is its velocity after 2 sec.? (See footnote, p. 51.)

27. In the motion studied under (d) of the present section, characterized by the equation $ma = m(g - g/k^2v^2)$, the coefficient of resistance $c = mg/k^2$ depends only upon the size and shape of the body, and not upon its mass. Using this fact, compare the ultimate speeds k_1 and k_2 acquired by a falling raindrop and a falling bullet of the same size, the specific gravity of lead being 11.3.

28. In problem 21 suppose the body experience a resistance equal to $2\,kv$ as well as the elastic force. Show that the motion becomes a dampened harmonic one of the type $s = Ae^{-kt}\sin(\mu t + B)$, A and B being constants of integration.

29. Study the motion of a particle repelled from a center by a force proportional to the distance of the particle from the center. Such a force might arise in studying electrified bodies.

30. A chain of length 5 ft. rests upon a smooth table with 1 ft. of its length hanging over the edge. Determine the motion of the chain as it slides off; also the time when it leaves the table and its velocity at this moment.

Hint. Treat the problem as if the chain were falling vertically, but only the weight of the part of the chain below the level of the table were effective in producing motion.

18. Motion of a particle in a plane. In taking up motion in the plane it is essential to recall that velocity, acceleration, and force are *no longer mere numbers*, but *directed magnitudes*, that is, *vectors*. We shall, as a rule, think of them as fixed by their two projections on the axes. We consider first a complete description of the motion (cf. p. 42). This will evidently be attained if we know the coördinates of the point at any given time; in other words, the complete description will consist in two equations:*

$$\left.\begin{array}{l} x = f(t) \\ y = g(t) \end{array}\right\}.$$

It is thus seen that the complete description of a plane motion consists in the description of two straight-line motions, namely, those of a point on the x-axis directly beneath (or above) the point in the plane, and of a point on the y-axis on a level with it. This dependence of the motion of a point in the plane upon two straight-line motions is fundamental, as our whole analytic treatment of plane motion depends upon it, and a similar statement holds for space.

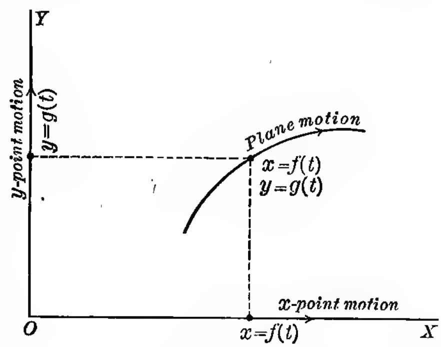


FIG. 19

The coördinates x and y of the moving point may be considered the projections of a *vector* whose beginning is at the origin of coördinates and whose end is at the moving point; we shall call it the *position vector*. We now *define* the velocity vector as *the derivative with respect to the time of the position vector*, and similarly we *define* the acceleration vector as *the derivative with respect*

* Another form of description sometimes of value consists in the equation of the path $f(x, y) = 0$, or $x = f(s)$, $y = g(s)$, together with a relation showing how the path is described, say $s = \phi(t)$, where s represents the length of the path measured from some convenient point on it. The question of passing from one kind of description to the other is left for the student to consider.

to the time of the velocity vector; * or, referring to §§ 5 and 6, the velocity vector V and the acceleration vector A are *defined* as follows by means of their projections, denoted in the usual manner:

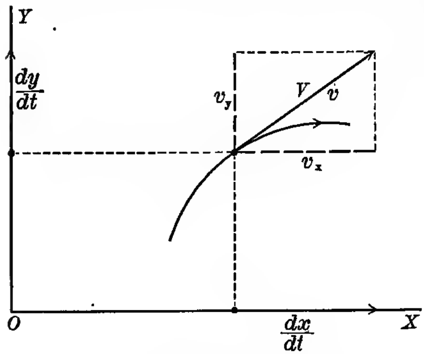


FIG. 20

$$\left.\begin{aligned} v_x &= \frac{dx}{dt} = f'(t) \\ v_y &= \frac{dy}{dt} = g'(t) \end{aligned}\right\},$$

$$\left.\begin{aligned} a_x &= \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = f''(t) \\ a_y &= \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = g''(t) \end{aligned}\right\};$$

and if f_x and f_y are the projections of the force vector F, we have

$$\left.\begin{aligned} ma_x &= f_x \\ ma_y &= f_y \end{aligned}\right\},$$

expressing analytically the fundamental relation between acceleration and force vectors. For the magnitudes of these vectors we have

$$v = \sqrt{v_x^2 + v_y^2},$$
$$a = \sqrt{a_x^2 + a_y^2},$$
$$f = \sqrt{f_x^2 + f_y^2}.$$

The magnitude of the velocity is called the *speed.* The essential distinction between speed, which is a positive number, and velocity, which is a vector, is one frequently overlooked, and serious errors result.

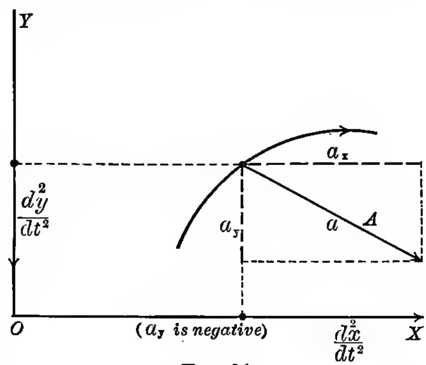


FIG. 21

Finally, we have for the direction of the velocity vector

* Note the close analogy of these definitions with the definitions of velocity and acceleration in a straight line (p. 42). The only difference is that *vectors* replace numbers.

$$\tan(x, v) = \frac{v_y}{v_x}, \qquad \left.\begin{aligned} \cos(x, v) &= -\sin(y, v) = \frac{v_x}{v} \\ \cos(y, v) &= +\sin(x, v) = \frac{v_y}{v} \end{aligned}\right\},$$

and similarly for the other vectors.

19. Some geometric properties of plane motion. In considering rectilinear motion our first concern was to fix a positive sense along the line. So in the plane curve, which is now to be our path, we fix a convenient point from which to measure the arc s, and fix a positive direction along the curve. We shall then denote by t the direction of the tangent to the curve at the point considered, the positive sense of its tangent agreeing with the positive sense of the curve. We shall denote by n the normal obtained by rotating the tangent through an angle $+\pi/2$. The terms *tangential velocity*, *normal velocity*, *tangential acceleration*, and *normal acceleration* will then readily be understood as the projections of the vectors named upon the directions named. We are now concerned with these vectors and with these projections.

(a) *The velocity vector is tangent to the path* (though it may be pointed in the negative sense of the tangent). For, from the calculus, τ being the angle between the tangent to the curve and the x-axis, $\tan\tau = (dy/dt) \div (dx/dt)$. But this is $v_y/v_x = \tan(x, v)$, so that $\tan\tau = \tan(x, v)$ and the theorem is true.

(b) *Tangential and normal velocities.* From the above it is at once evident that *the normal velocity is zero.* For the tangential velocity we have, referring to IV, § 5, $v_t = v_x \cos(x, t) + v_y \cos(y, t)$. But in the calculus it is shown that

$$\cos(x, t) = \cos\tau = \frac{dx}{ds} = \frac{dx}{dt} \div \frac{ds}{dt},$$

$$\cos(y, t) = \frac{dy}{dt} \div \frac{ds}{dt},$$

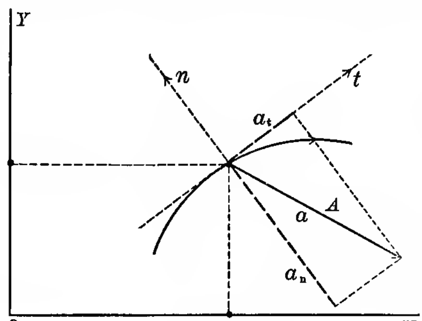
and that

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

Using these values with $v_x = dx/dt$ and $v_y = dy/dt$, we obtain

$v_t = ds/dt$; that is, *the tangential velocity is the derivative of the arc with respect to the time.**

(c) *The acceleration vector is not, in general, tangent to the path.* For this would mean $a_y/a_x = v_y/v_x$, or $(dv_y/dt) \div (dv_x/dt) = v_y/v_x$; i.e. $dv_y/v_y = dv_x/v_x$, and integrating, $\log v_y = \log v_x + \log m$, whence $v_y = mv_x$ or $dy/dt = m(dx/dt)$, and integrating again, $y = mx + c$. We therefore have the theorem: *If the acceleration vector is continually tangent to the path, the path is straight.* As a corollary we see that where the path is curved the acceleration is *not* tangent to the path.



(a_n *is negative*)

FIG. 22

(d) *The tangential and normal accelerations.* These, like the tangential velocity, we obtain by application of IV, § 5. First

$$\begin{aligned} a_t &= a_x \cos(x, t) + a_y \cos(y, t) \\ &= a_x \frac{dx}{ds} + a_y \frac{dy}{ds} = \left(a_x \frac{dx}{dt} + a_y \frac{dy}{dt}\right) \div \frac{ds}{dt} \\ &= \left(\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} + \frac{dy}{dt}\cdot\frac{d^2y}{dt^2}\right) \div \frac{ds}{dt} = \left\{\frac{1}{2}\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]\right\} \div \frac{ds}{dt} \\ &= \left[\frac{1}{2}\frac{d}{dt}\left(\frac{ds}{dt}\right)^2\right] \div \frac{ds}{dt} = \frac{d^2s}{dt^2}, \end{aligned}$$

that is, *the tangential acceleration is the second derivative of the arc with respect to the time.* For the normal acceleration

$$\begin{aligned} a_n &= a_x \cos(x, n) + a_y \cos(y, n) = -a_x \cos(y, t) + a_y \cos(x, t) \\ &= -a_x \frac{dy}{ds} + a_y \frac{dx}{ds} = \left(a_y \frac{dx}{dt} - a_x \frac{dy}{dt}\right)\frac{dt}{ds} = (a_y v_x - a_x v_y) \div \frac{ds}{dt}. \end{aligned}$$

* Notice that the tangential velocity may be positive or negative according as the point is moving in the positive or negative direction along the curve. The speed is the (*positive*) numerical value of the tangential velocity.

To obtain an interpretation for this we refer again to the calculus, where we find for the radius of curvature the expression

$$R=\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}\div\left(\frac{d^2y}{dt^2}\cdot\frac{dx}{dt}-\frac{d^2x}{dt^2}\cdot\frac{dy}{dt}\right)=\left(\frac{ds}{dt}\right)^3\div(a_yv_x-a_xv_y),$$

whence $$a_yv_x-a_xv_y=\left(\frac{ds}{dt}\right)^3\div R.$$

We have, therefore, $$a_n=\left(\frac{ds}{dt}\right)^2\div R=\frac{v^2}{R},$$

or, *the normal acceleration is the square of the speed divided by the radius of curvature.*

The usual formula for the radius of curvature is

$$R=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}\div\frac{d^2y}{dx^2},$$

and the values $$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$$

and $$\frac{d^2y}{dx^2}=\left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]\div\frac{dx}{dt}=\left(\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}\right)\div\left(\frac{dx}{dt}\right)^3$$

reduce this to the form given.

In the numerator of R occurs a square root, and the reduction made in deriving the expression for a_n amounts to giving to this square root the sign of ds/dt. The result is that R must be considered positive or negative according as the center of curvature lies in the positive direction of the normal to the curve as above fixed, or in the negative direction. The formulas found in (b) and (d) are important, and are therefore gathered together here:

$$v_t=\frac{ds}{dt},\quad v_n=0,\quad a_t=\frac{d^2s}{dt^2},\quad a_n=\frac{v^2}{R}.$$

The two projections of the acceleration are interesting in connection with their effect. The formulas show that the effect of the tangential acceleration is exclusively to change the tangential velocity; the effect of the normal acceleration is exclusively to change the curvature of the path. It is easily seen from physical considerations that the normal component of the acceleration points from the curve *toward* the center of curvature.

IX. PROBLEMS ON PLANE MOTION

In the following problems derive, when possible, the equation of the path by eliminating t from the parameter equations; find also expressions for v_x, v_y and a_x, a_y. Plot the path, and draw in different-colored inks for various values of t the velocity vector (red, say) and the acceleration vector

(blue, say), and the tangential and normal components of the latter (blue dotted, say). Add a description of any salient characteristics of the motion. A different scale may be used for position vector, velocity vector, and acceleration vector in case any become too large or too small to yield a clear figure, but in this case the scales should be clearly designated.

1. $\left.\begin{aligned} x &= t^2 \\ y &= t^3 \end{aligned}\right\}.$

Solution. The path is seen to be $y^2 = x^3$, the "semicubical parabola." By differentiation we find $v_x = 2\,t$, $v_y = 3\,t^2$, and $a_x = 2$, $a_y = 6\,t$. We next proceed to draw the path and the components parallel with the axes of velocity and the total velocity, first making a table (see Fig. 23). We may always compute the magnitude of the velocity, or the *speed:*

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{4\,t^2 + 9\,t^4} = |t|\sqrt{4 + 9\,t^2}.$$

(The vertical bars about t here denote that the *positive* numerical value of t is to be taken.) We see from this that the speed vanishes for $t = 0$ and increases with $|t|$. As $v_y = 3\,t^2$ is always positive except for $t = 0$, it follows that the motion is always upward.

We next turn to a consideration of the acceleration. The components may be drawn as before for the velocity, being previously tabulated (see Fig. 24). The resultant should then be drawn and a tangent and normal. The tangent should be drawn by means of its slope, and not merely by a ruler laid along the curve. The tangential and normal accelerations are the projections on tangent and normal of the acceleration vector.

Remark. Analytic expressions for tangential and normal acceleration may be calculated by the formulas given on p. 59. The only difficulty consists in determining the sign of the radical in $ds/dt = v_t = \sqrt{(dx/dt)^2 + (dy/dt)^2}$. In our present problem if we measure s, say, from the origin, and always increasing with y, then as y always increases with t, so also does s, and ds/dt is positive or zero. Hence

$$v_t = \frac{ds}{dt} = +\sqrt{(2\,t)^2 + (3\,t^2)^2} = |t|\sqrt{4 + 9\,t^2},$$

and $a_t = \dfrac{a_x v_x + a_y v}{v_t} = \dfrac{4\,t + 18\,t^3}{|t|\sqrt{4 + 9\,t^2}}$, and $a_n = \dfrac{-\,a_x v_y + a_y v_x}{v_t} = \dfrac{12\,t^2}{|t|\sqrt{4 + 9\,t^2}}$.

2. $\left.\begin{aligned} x &= t^2 \\ y &= 2\,t \end{aligned}\right\}.$ 3. $\left.\begin{aligned} x &= t \\ y &= t^3 \end{aligned}\right\}.$ 4. $\left.\begin{aligned} x &= 1 + \cos t \\ y &= 2\cos(t/2) \end{aligned}\right\}.$ 5. $\left.\begin{aligned} x &= t \\ y &= 1/(1 + t^2) \end{aligned}\right\}.$

Take for t in problem 4 a series of values $0, \pi/3, 2\,\pi/3, \pi, \cdots$.

6. $\left.\begin{aligned} x &= t \\ y &= 2\sqrt{1 - t^2} \end{aligned}\right\}.$

8. $\left.\begin{aligned} x &= \cos t \\ y &= 2\sin t \end{aligned}\right\}.$

7. $\left.\begin{aligned} x &= \cos kt \\ y &= \sin kt \end{aligned}\right\}.$

9. $\left.\begin{aligned} x &= (1 - t^2)/(1 + t^2) \\ y &= 2\,t/(1 + t^2) \end{aligned}\right\}.$

The path is closed. How long does it take for the body to make a complete circuit of its path?

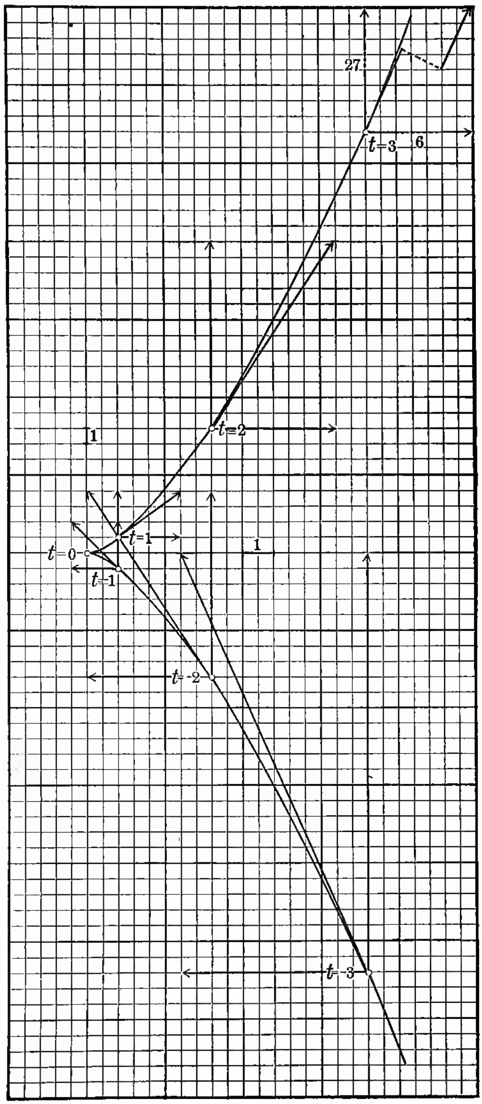


FIG. 23 (Velocities)

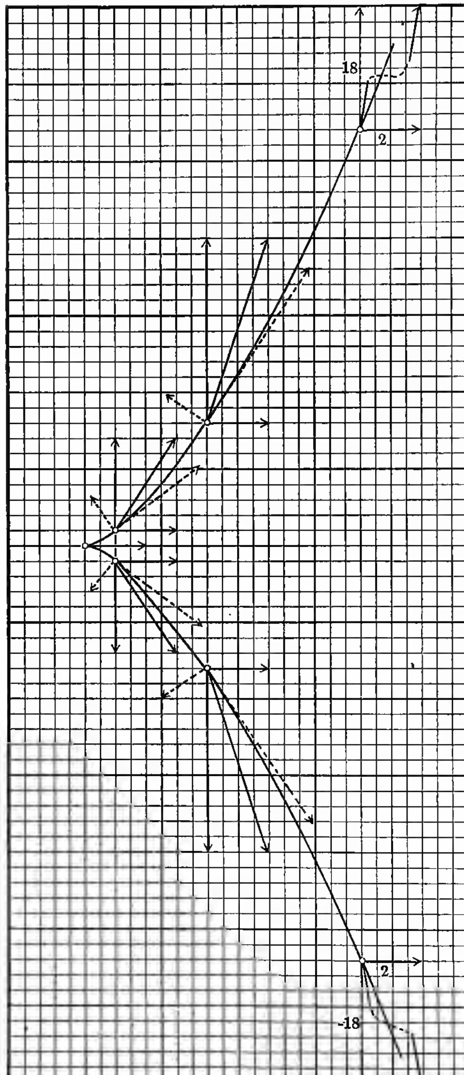


Fig. 24 (Accelerations)

10. $\left.\begin{aligned} x &= t \\ y &= \sin t \end{aligned}\right\}.$

11. $\left.\begin{aligned} x &= \log(t + \sqrt{1+t^2}) \\ y &= \sqrt{1+t^2} \end{aligned}\right\}.$

Show that the path is a catenary $y = \frac{1}{2}(e^x + e^{-x})$ and that it is described with constant speed.

12. Show that the normal acceleration is the product of the tangential velocity by the rate of change of direction of motion, i.e. $a_n = (ds/dt) \cdot (d\tau/dt)$ where $\tau = \arctan v_y/v_x$.

20. The differential equations of plane motion. The differential equations of plane motion were expressed in the last paragraph in the relation between force and acceleration. Usually they are written

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= f_x \\ m\frac{d^2y}{dt^2} &= f_y \end{aligned}\right\}. \quad 1$$

The problem of determining the "complete description" from these equations together with proper auxiliary conditions is one whose difficulty depends entirely upon the nature of the force. The simplest case is that in which the component f_x depends only upon x and dx/dt, and f_y only upon y and dy/dt. In this case the solution of each differential equation is a separate problem. Usually, however, both f_x and f_y depend upon both x and y and upon v_x and v_y. The equations are therefore simultaneous differential equations, and are solved by various devices depending upon the special problems attacked, important among which is the elimination of one of the dependent variables.

If we know the path in advance (see footnote, p. 55), we only need determine the function $s = \phi(t)$ giving the way in which the path is described. To this end we use instead of the differential equations (1) the single equation which comes from considering tangential components (see § 6):

$$m\frac{d^2s}{dt^2} = f_s.$$

21. Some special plane motions.

(a) *Circular motion.* Let θ denote the angle between the x-axis and a line joining the origin to the point (x, y), which we suppose

to move on a circle of radius r about the origin. Let us count the arc s of this circular path from the point where it crosses the x-axis. Then $x = r\cos\theta$, $y = \sin\theta$, and $s = r\theta$. From the preceding section we then have, by differentiating both members of the equation $s = r\theta$,

$$v_t = \frac{ds}{dt} = r\frac{d\theta}{dt}, \qquad a_t = \frac{d^2s}{dt^2} = r\frac{d^2\theta}{dt^2},$$

and

$$a_n = \left(\frac{ds}{dt}\right)^2 \div r = r^2\left(\frac{d\theta}{dt}\right)^2 \div r = r\left(\frac{d\theta}{dt}\right)^2.$$

The quantities $\omega = d\theta/dt$ and $\alpha = d^2\theta/dt^2$ are called respectively the *angular velocity* and the *angular acceleration*, and are *the time rates of change of the angle and of the angular velocity* ω. These notions have also an important application to the study of the rotation of a rigid body about an axis (see p. 91).

As an illustration of the general method for obtaining projections upon given lines of the velocity and acceleration vectors, we shall derive these results directly, obtaining incidentally an interesting verification of our general theory.

Using IV of § 5, we have $v_t = v_x\cos(x, t) + v_y\cos(y, t)$. But as $(x, t) = \theta + \pi/2$ and $(y, t) = \theta$, $\cos(x, t) = -\sin\theta$ and $\cos(y, t) = \cos\theta$; moreover

$$v_x = \frac{dx}{dt} = -r\sin\theta\frac{d\theta}{dt}$$

and

$$v_y = \frac{dy}{dt} = r\cos\theta\frac{d\theta}{dt}.$$

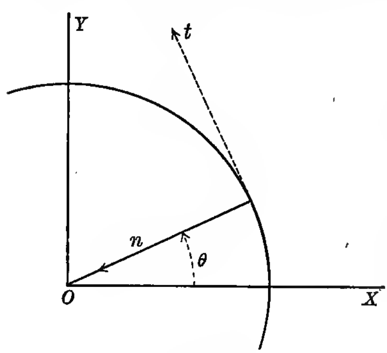


FIG. 25

Using these values, we have

$$v_t = r(\sin^2\theta + \cos^2\theta)\frac{d\theta}{dt} = r\frac{d\theta}{dt}.$$

Again, by differentiating v_x and v_y, we have

$$a_x = -r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\left(\frac{d^2\theta}{dt^2}\right),$$

$$a_y = -r\sin\theta\left(\frac{d\theta}{dt}\right)^2 - r\cos\theta\left(\frac{d^2\theta}{dt^2}\right).$$

Forming now a_t after the manner of v_t, we have $a_t = r(d^2\theta/dt^2)$; and forming a_n, keeping in mind that $(x, n) = \theta + \pi$, $(y, n) = \theta + \pi/2$, so that $\cos(x, n) = -\cos\theta$, $\cos(y, n) = -\sin\theta$, we have $a_n = r(d\theta/dt)^2$, all these values agreeing with those found above. The results may be stated: *The tangential velocity is r times the angular velocity; the tangential acceleration is r*

times the angular acceleration; and the normal acceleration is r times the square of the angular velocity.

Ex. 1. If ω is constant, $\theta = \omega t$. Using the equations $x = r\cos\omega t$, $y = r\sin\omega t$, find directly the values for v_t and a_n, and show that $a_t = 0$. Show that the force producing the motion is always toward the center (called centripetal force). The point on the x-axis given by $x = r\cos\omega t$ is the projection of the moving point. Its motion is simple harmonic motion (see p. 48).

(b) *Projectiles.* If a body is thrown into the air and left to the action of gravity alone, resistance of the air and the effect of the rotation of the earth being neglected, the differential equations become

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= 0 \\ m\frac{d^2y}{dt^2} &= -mg \end{aligned}\right\}, \quad \text{or} \quad \left.\begin{aligned} \frac{d^2x}{dt^2} &= 0 \\ \frac{d^2y}{dt^2} &= -g \end{aligned}\right\}.$$

If we take for origin of coördinates the point from which the body was thrown, and if the body was thrown in a direction making an

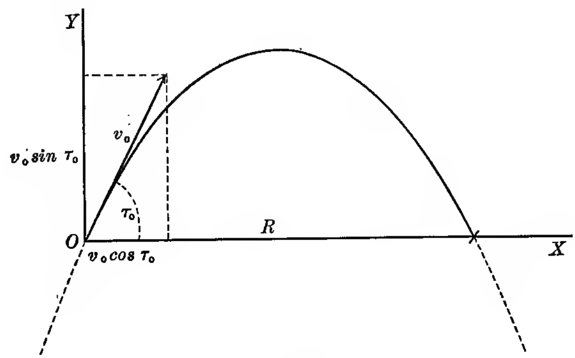


Fig. 26

angle τ_0 with the horizontal with an initial speed v_0, the auxiliary conditions are $x = 0$, $y = 0$; $v_x = v_0\cos\tau_0$, $v_y = v_0\sin\tau_0$ for $t = 0$. A first integration of the differential equations gives dx/dt, or $v_x = c_1$; dy/dt, or $v_y = -gt + c_2$. When $t = 0$ these reduce to $v_0\cos\tau_0$ and $v_0\sin\tau_0$ respectively. Hence $c_1 = v_0\cos\tau_0$ and $c_2 = v_0\sin\tau_0$. Hence

$dx/dt = v_0 \cos\tau_0$ and $dy/dt = -gt + v_0 \sin\tau_0$. Integrating again, we have

$$x = (v_0 \cos\tau_0)t + c_3, \quad y = -\tfrac{1}{2}gt^2 + (v_0 \sin\tau_0)t + c_4.$$

As $x = 0$ and $y = 0$ where $t = 0$, $c_3 = 0$ and $c_4 = 0$. Hence we have the complete description

$$\left.\begin{array}{l} x = (v_0 \cos\tau_0)t \\ y = -\tfrac{1}{2}gt^2 + (v_0 \sin\tau_0)t \end{array}\right\}.$$

Some important results in the theory of projectiles are now left for the student to derive in the ensuing exercises and in some later problems.

Ex. 2. Verify, by eliminating t, that the equation of the *path* of the projectile is

$$y = \frac{-gx^2}{2(v_0 \cos\tau_0)^2} + (\tan\tau_0)x.$$

What is the nature of this curve? Draw it for $\tau_0 = 45°$, $v_0 = 40$ ft. per sec., using $g = 32$. Draw the velocity vectors for $t = 0$, $t = \frac{1}{2}$, $t = 1$, $t = \frac{3}{2}$.

Ex. 3. *Range*. By the *range*, R, of a projectile is meant the horizontal distance covered before it reaches the ground again. Show that

$$R = \frac{v_0^2 \sin 2\tau_0}{g}.$$

What elevation gives the greatest range? Show that any two values of τ_0 differing by equal amounts on either side of this elevation give the same range.

Ex. 4. Find the *time of flight*. The initial speed v_0 being given, what elevation gives the greatest time of flight?

Ex. 5. Suppose the initial speed v_0 be given. Then for various values of τ_0 we have a system of parabolas with vertical axes and all passing through the same point. Any point in the plane which is covered by this system is *within range*, that is, can be hit with the given initial velocity. In other words, all points within the envelope of the system are within range. Find the envelope, writing the system in the form $y = -gx^2(1 + m^2)/2v_0^2 + mx$. The result should be a parabola whose highest point is that of the highest path of the system and whose breadth at the level of the initial point is twice the maximum range. Show that your result agrees with these statements.

Ex. 6. Refer the path of the projectile of Ex. 2 to its vertex by a shift of the axes (see Analytic Geometry). What are the coördinates of the vertex? What is the latus rectum? Show that all the trajectories corresponding to a given initial speed have the same directrix.

(c) *Pendulum*. We are dealing with a particle moving in a circular arc under gravity. Knowing the path, we proceed as

indicated in § 20, except that instead of s it will be convenient to use the angle between the pendulum and a line vertically downward. Then $s = l\theta$, l denoting the length of the pendulum; and as the tangential component of the force is $-mg \sin \theta$, we have

$$m \frac{d^2 s}{dt^2} = ml \frac{d^2 \theta}{dt^2} = -mg \sin \theta,$$

or $$\frac{d^2 \theta}{dt^2} = -\frac{g}{l} \sin \theta.$$

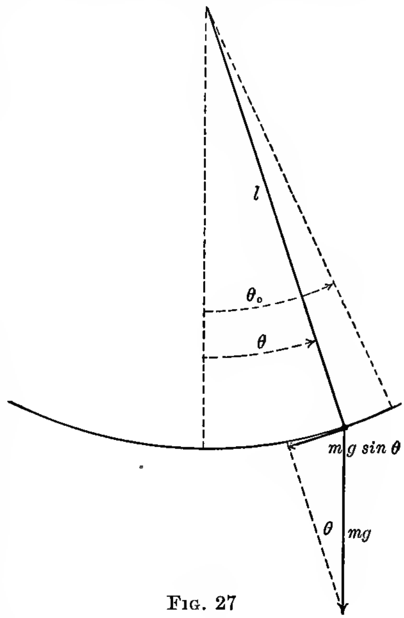


FIG. 27

To avoid difficulties in integration it is usual to substitute for $\sin \theta$ on the right the quantity θ, which approximation * is in error by less than $\theta^3/6$, where θ is measured in radians, and which is quite satisfactory when θ does not exceed a twentieth of a radian. The resulting equation,

$$\frac{d^2 \theta}{dt^2} = -\frac{g}{l} \cdot \theta,$$

is easily integrated (see (b), § 16), and yields

$$\theta = A \sin\left(\sqrt{\frac{g}{l}} \cdot t\right) + B \cos\left(\sqrt{\frac{g}{l}} \cdot t\right).$$

* If this approximation is not used, the method is as follows: Multiplying by $d\theta/dt$ (see (c), § 16), the equation becomes

$$\frac{d}{dt} \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \frac{g}{l} \cdot \frac{d}{dt} \cos \theta, \quad \text{whence} \quad \left(\frac{d\vartheta}{dt}\right)^2 = \frac{2g}{l} \cos \theta + c_1.$$

Let the pendulum start from an angle θ_0, so that $d\vartheta/dt = 0$ when $\theta = \theta_0$; then $c_1 = -(2g/l) \cos \theta_0$, and we have

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l} (\cos \theta - \cos \theta_0) \quad \text{and} \quad \sqrt{\frac{2g}{l}}\, t + c_2 = \int \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}}.$$

This integral cannot be expressed by means of elementary functions, and integration by series or by means of elliptic integrals must be resorted to. In order to identify this with an elliptic integral, we must reduce it to a standard form by substitutions (see B. O. Peirce, *A Short Table of Integrals*). First write $\cos \theta = 1 - 2 \sin^2 (\theta/2)$,

If we count the time so that $t = 0$ when $\theta = 0$, we find $B = 0$, so that the solution becomes

$$\theta = A \sin\left(\sqrt{\frac{g}{l}} \cdot t\right).$$

The motion is thus oscillatory. The period, $2\pi\sqrt{l/g}$, or the time of a complete swing to and fro, is seen to be independent of A, i.e. of the length of the swing. As T and l are observable, the pendulum gives a useful method of determining the value of g. The motion in the circular arc is *simple harmonic* for

$$s = \frac{\theta}{l} = \frac{A}{l} \sin\left(\sqrt{\frac{g}{l}} \cdot t\right).$$

(d) *Motion under central forces.* By a *central force* is usually understood one whose direction is always in the line joining the particle with a fixed point, or center, and whose magnitude depends only upon the distance of the particle from this center. We shall consider first a special case, namely that in which the force is always directed *toward* the center and whose magnitude varies with the distance from it. Then $f = mk^2r$, where $r =$

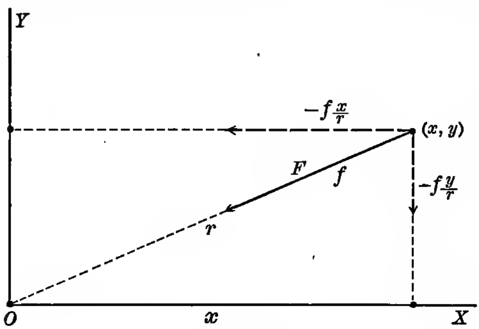


Fig. 28

$\cos\theta_0 = 1 - \sin^2(\theta_0/2)$; then, introducing a new variable ϕ, by the relation $\sin\theta/2 = \sin(\theta_0/2)\sin\phi$, we obtain

$$\int\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \sqrt{2}\int\frac{d\phi}{\sqrt{1 - \sin^2\left(\frac{\theta_0}{2}\right)\mathrm{siu}^2\phi}} = \sqrt{2}\,F\left(\phi, \sin\frac{\theta_0}{2}\right).$$

If $t = 0$ when $\theta = 0$, and hence when $\phi = 0$, we have, since $F\left(0, \sin\frac{\theta_0}{2}\right) = 0$,

$$t = \frac{l}{g}F\left(\phi, \sin\frac{\theta_0}{2}\right).$$

When $\theta = \theta_0$, $\phi = \pi/2$, so that the whole time of an oscillation T, being four times the time consumed as θ goes from 0 to θ_0, is $4\,(l/g)\,F[(\pi/2), \sin(\theta_0/2)]$. As an illustration the student may show that the period of a pendulum swinging through an angle 60° to either side of the vertical is $6.744\sqrt{l/g}$. A further study of the properties of elliptic integrals is recommended at this point to students who find the subject interesting (see Byerly, *Integral Calculus*).

$\sqrt{x^2+y^2}$, the force center being taken as origin, and $\cos(x, F) = -x/r$, $\cos(y, F) = -y/r$, so that $f_x = -mk^2x$ and $f_y = -mk^2y$. The differential equations of motion therefore take the form

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -k^2x \\ \frac{d^2y}{dt^2} &= -k^2y \end{aligned}\right\}.$$

Referring to (b), § 17, we see that the projections of the motion on the axes are both *simple harmonic motion.* To find the path, we endeavor to eliminate t from the solution

$$\left.\begin{aligned} x &= c_1 \cos kt + c_2 \sin kt \\ y &= c_3 \cos kt + c_4 \sin kt \end{aligned}\right\}.$$

This is best done by solving for $\sin kt$ and $\cos kt$ and squaring and adding. We find *

$$\cos kt = \frac{c_4x - c_2y}{c_1c_4 - c_2c_3}, \quad \sin kt = \frac{-c_3x + c_1y}{c_1c_4 - c_2c_3},$$

whence $$(c_4x - c_2y)^2 + (-c_3x + c_1y)^2 = (c_1c_4 - c_2c_3)^2.$$

This is a conic section referred to its center, for it is of the second degree and the linear terms are absent. That it is an ellipse is easily verified by the test given in Analytic Geometry, namely that $B^2 - 4AC < 0$ where A, B, and C are the coefficients of x^2, xy, and y^2 respectively. The motion is therefore called *elliptic harmonic motion.*

Ex. 7. Show that the force producing the motion $x = a \cos kt$, $y = b \sin kt$, is toward the origin and varies with the distance from the origin.

Returning to the general problem of central motion, we shall have the same direction cosines for the force, but the magnitude f will be some function of r, say, $f = \psi(r)$. The equations become

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \psi(r)\frac{x}{r} \\ m\frac{d^2y}{dt^2} &= \psi(r)\frac{y}{r} \end{aligned}\right\}.$$

* Unless $c_1c_4 - c_2c_3 = 0$. The motion in this case is simple harmonic motion in a straight line, as the student may show as an exercise.

These equations present the difficulty mentioned in § 20, that both equations contain both variables, since $r=\sqrt{x^2+y^2}$. Two integrals may, however, be obtained,— first, the *integral of areas.* Multiplying the first equation by $-y$ and the second by $+x$ and adding, we have

$$-y\frac{d^2x}{dt^2}+x\frac{d^2y}{dt^2}=0,$$

that is, $$\frac{d}{dt}\left(x\frac{dy}{dt}-y\frac{dy}{dt}\right)=0, \quad \text{or} \quad x\frac{dy}{dt}-y\frac{dx}{dt}=c_1.$$

The meaning of this is clearer upon using polar coördinates, when it becomes

$$r\cos\theta\left(\frac{dr}{dt}\sin\theta+r\cos\theta\frac{d\theta}{dt}\right)-r\sin\theta\left(\frac{dr}{dt}\cos\theta-r\sin\theta\frac{d\theta}{dt}\right)=c_1,$$

or $$\frac{r^2d\theta}{dt}=c_1.$$

Now let A represent the area swept out by the moving radius vector. We have, by the calculus,

$$A=\tfrac{1}{2}\int r^2d\theta, \quad \text{or} \quad dA=\tfrac{1}{2}r^2d\theta, \quad \text{so that} \quad \frac{r^2d\theta}{dt}=\frac{2\,dA}{dt},$$

and the equation becomes

$$\frac{dA}{dt}=\frac{c_1}{2}, \quad \text{so that} \quad A=\frac{c_1}{2}t+c_2.$$

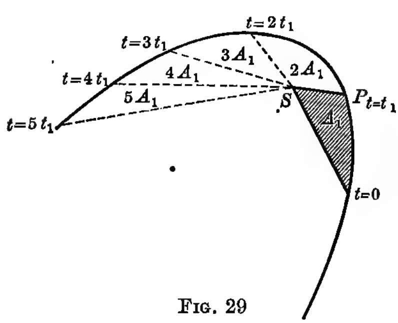


Fig. 29

If we count the area from the time $t=0$, we have $c_2=0$, and we may say *the area swept out by the radius vector in the case of a particle moving under a central force is proportional to the time consumed.*

The second integral, known as the *energy integral,* results from observing that $\partial r/\partial x=x/r$ and $\partial r/\partial y=y/r$, as may be seen by partial differentiation of the equation $r^2=x^2+y^2$. The equations of motion thus become

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \psi(r)\left(\frac{\partial r}{\partial x}\right) = -\frac{\partial}{\partial x}\Psi(r) \\ m\frac{d^2y}{dt^2} &= \psi(r)\left(\frac{\partial r}{\partial y}\right) = -\frac{\partial}{\partial y}\Psi(r) \end{aligned}\right\},$$

where $\Psi(r)$ is the negative of the integral with respect to r of $\psi(r)$. Multiplying by dx/dt and dy/dt and adding, we have

$$m\left(\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2}\right) = -\frac{\partial\Psi(x)}{\partial x}\cdot\frac{dx}{dt} - \frac{\partial\Psi(r)}{\partial y}\cdot\frac{dy}{dt}$$

or

$$\frac{d}{dt}\cdot\frac{mv^2}{2} = -\frac{d\Psi(r)}{dt},$$

whence

$$\tfrac{1}{2}mv^2 + \Psi(r) = c_3.$$

The first term, $\frac{1}{2}mv^2$, is the *kinetic energy* (see § 25) of the particle, and the second, $\Psi(r)$, is called the *potential energy* (see § 24); the equation states that *the sum of the kinetic and potential energies of the particle is constant.* This is a special case of an important principle in mathematical physics, known as the *conservation of energy.*

We consider now briefly the motion of the planets about the sun. If we consider these bodies as spheres, the forces they exert on each other may be regarded as acting at their centers, as is shown in the theory of attraction. Let us take the sun at the origin and consider one of the planets. It will be attracted (see (c), § 17) with a central force whose magnitude is proportional to the reciprocal of the square of its distance away, say, k/r^2. The force being attractive, we have

$$\psi(r) = -\frac{k}{r^2} \quad \text{and} \quad \Psi(r) = +\int\frac{k}{r^2}dr = -\frac{k}{r}.$$

The force exerted by the other planets is small in comparison with that exerted by the sun, so we shall get an approximation to the motion if we neglect it. The energy integral becomes $\frac{1}{2}mv^2 - k/r = c_3$, or, using polar coördinates, since

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2,$$

$$\frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] - \frac{k}{r} = c_3.$$

We may now obtain the differential equation of the path if we can eliminate t. This may be done by the integral of areas, which may be written $d\theta/dt = c_1/r^2$. We have then, since $dr/dt = (dr/d\theta)\cdot(d\theta/dt)$,

$$\frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right] - \frac{k}{r} = \frac{1}{2}m\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\left(\frac{d\theta}{dt}\right)^2 - \frac{k}{r},$$

so that the differential equation becomes

$$\frac{1}{2}m\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\frac{c_1^2}{r^4} - \frac{k}{r} = c_3.$$

This, solved for $dr/d\theta$, yields

$$\frac{dr}{d\theta} = \sqrt{\frac{2\,r^4}{mc_1^2}\left(c_3 + \frac{k}{r}\right) - r^2} = r\sqrt{\frac{2\,c_3}{mc_1^2}r^2 + \frac{2\,k}{mc_1^2}r - 1}.$$

Whence, integrating, $\theta = \displaystyle\int \frac{dr}{r\sqrt{\frac{2\,c_3}{mc_1^2}r^2 + \frac{2\,k}{mc_1^2}r - 1}} + c_4.$

This integral may be evaluated by a substitution $r = 1/u$, or by reference to tables (e.g. Peirce's *Short Table of Integrals*, No. 183), which gives

$$\theta - c_4 = \arcsin\left(\frac{\frac{2\,k}{mc_1^2}r - 2}{r\sqrt{\frac{4\,k^2}{m^2c_1^4} + \frac{8\,c_3}{mc_1^2}}}\right) = \arcsin\left(\frac{k - \frac{mc_1^2}{r}}{\sqrt{k^2 + 2\,mc_1^2c_3}}\right).$$

This may be written

$$\sin(\theta - c_4) = \frac{k - \frac{mc_1^2}{r}}{\sqrt{k^2 + 2\,mc_1^2c_3}},$$

or if we suppose $c_4 = -\pi/2$, which amounts only to a particular choice of the direction of the x-axis, we have

$$\sin(\theta - c_4) = \sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta,$$

and the equation may be written

$$\frac{\left(\frac{mc_1^2}{k}\right)}{r} = 1 - \sqrt{1 + \frac{2\,mc_1^2c_3}{k^2}}\cos\theta.$$

Comparing this with the equation (see Analytic Geometry) $l/r = 1 - e\cos\theta$, we see that *the path is a conic section referred to its focus as origin.* This fact was discovered by observation by Kepler, and bears the name Kepler's first law. The eccentricity is

$$e = \sqrt{1 + \frac{2\,mc_1^2 c_3}{k^2}},$$

and is $\gtreqless 1$, according as $2\,mc_1^2c_3/k \gtreqless 0$, that is, according as $c_3 \gtreqless 0$. But, referring to the energy integral, we find an interpretation for c_3. For if we denote by v_0 and r_0 the speed of the planet and its distance from the sun when $t = 0$, we have, since the energy equation holds for every t, $c_3 = \frac{1}{2}mv_0^2 - (k/r_0)$. Hence $e \gtreqless 1$ according as $c_3 \gtreqless 0$, that is, according as $v_0^2 \gtreqless 2\,k/mr_0$. Thus the path will be a parabola if the initial speed is $\sqrt{2\,k/mr_0}$, and this no matter what the initial direction of the planet. For larger initial speeds the paths are hyperbolas, for smaller ones ellipses. Planets move only in ellipses. Comets have orbits of any of the types. The earth's orbit is very nearly a circle, its eccentricity is about $\frac{1}{60}$.

The motion of the body in its path is determined by the integral of areas. Equal areas being swept out in equal times, it follows that the planets move fastest when nearest the sun. This result may also be read from the energy integral.

X. PROBLEMS ON BODIES MOVING IN A PLANE UNDER THE ACTION OF FORCES

1. A snowball fight is arranged between two boys who can throw with speeds of 78 ft. per sec. and 83 ft. per sec. respectively. In order to equalize the contest the boy with the less speed is given a station $6\frac{1}{4}$ ft. higher than that of his comrade. Assuming that the effect of the snowball varies with the square of the speed, does this arrangement equalize the fight?

Solution. We must first find the expression for the speed of a projectile. Taking the equations of motion, finding v_x and v_y, and squaring and adding, we have $v^2 = v_0^2 - 2\,gt\,(v_0 \sin\tau_0) + g^2t^2$. We wish to know the value of this expression for a given value of y. To do this it is only necessary to find t from the equation for y, $y = -gt^2/2 + (v_0 \sin\tau_0)t$, and substitute it in the equation for v^2, or, in other words, to eliminate t. This is easily done by multiplying the second equation by $2\,g$ and adding. The result is $v^2 + 2\,gy = v_0^2$, or $v^2 = v_0^2 - 2\,gy$. Now let v' be the speed with which the stronger boy

hits the weaker at a height $6\frac{1}{4}$ ft. above him; then $v'^2 = (83)^2 - 2g(6\frac{1}{4})$; and if v'' be the speed with which the weaker boy hits the stronger at a height $6\frac{1}{4}$ ft. below him, then $v''^2 = (78)^2 + 2g(6\frac{1}{4})$. Computing these with $g = 32.2$, we find $v'^2 = v''^2 = 6486.5$, that is, the handicap was correct.

Remark. The striking circumstance in this problem is that in the elimination of t between the expressions for y and v^2, τ_0 drops out, and we find that with a given initial velocity the speed depends only on the height. This fact greatly simplifies the solution.

2. If a man can throw a baseball 350 ft., with what speed does it leave his hand?

3. Suppose a rifle ball in traveling 100 yd. deviates from a horizontal line by less than 1 in. Show that its initial speed must exceed 2070 ft. per sec.

4. What elevation will give a range of 5000 yd. with an initial speed of 1500 ft. per sec.?

5. Show that the heights to which a body will rise when projected with a given initial speed, but with elevations 30°, 45°, 60°, and 90°, will be in the ratios 1 : 2 : 3 : 4.

6. A man who can make a standing broad jump of 8 ft. stands at the window of a burning building 60 ft. above the level of a river whose depth at any point is one tenth its distance from the face of the building. How deep water can he reach by jumping? If he dived, at what angle would he strike the water? (Solve this problem on the hypothesis that he jumps into the air at an angle 45°. As a matter of fact an angle $\frac{1}{2}$ arccos $\frac{15}{17}$ would be more advantageous in both respects, as may be shown by the student who is sufficiently interested.)

7. What area of the surface of a building can the stream from a fire hose cover, the nozzle being 3 m. from the building and the speed of the stream being 20 m. per sec.? (See Ex. 5 under Projectiles, § 21.)

8. Show that the point (x_1, y_1) is within range if $y_1 + \sqrt{x_1^2 + y_1^2} \leqq v_0^2/g$. *Hint.* Use Ex. 5.

9. Show that the area of that part of a level plane which is within range of a gun a height h above the plane increases linearly with h, and has the form $A + 2h\sqrt{\pi A}$ where A is the area within range when the gun is in the plane. Show that the area within range varies as a biquadratic function of the initial speed.

10. Show that the speed of a projectile is the same at any two points of the same height. Show that the initial speed is the speed that a body would acquire in falling from rest at the directrix of the parabola to the level of the initial point. Extend the reasoning so as to prove a similar theorem for the speed at any point.

11. A gun is at the bottom of a hill the angle of inclination of whose face is i. Show that the range for an elevation τ_0 is

$$R = \frac{2\,v_0^2}{g} \cdot \frac{\cos\tau_0 \sin(\tau_0 - i)}{\cos^2 i},$$

that the best range is obtained when the initial velocity vector bisects the angle between the face of the hill and the vertical, and that this best range is $R = v_0^2/g(1 + \sin i)$.

12. The focus of the trajectory of a projectile is above or below the horizontal plane from which it was projected according as the angle of elevation is greater or less than the elevation giving the maximum range.

13. Water jets are sent out in all directions with a speed of 30 ft. per sec. from a fountain, the nozzle being in the form of a pierced ball. What is the nature and the equation of the surface of the water?

14. Show that when a number of projectiles are fired simultaneously from a point in directions which all lie in the same plane, the projectiles will, at any instant, all lie in a plane parallel to the plane of the initial velocities. Resolve the acting forces and initial velocities into components normal to and parallel with the plane. Show that the bodies will all move the same distance normal to the plane.

15. Show that if three bodies are projected simultaneously from the same point and in the same vertical plane, the triangle subtended by them remains similar to itself and has an area varying with the square of the time.

16. A clock gains 3 min. per day. What proportion of length should the pendulum be lengthened or shortened?

Use the results of the approximation for small arcs.

17. A pendulum making $2n$ swings per day is lengthened from l to $l + \Delta l$. Show that it will lose approximately $n\Delta l/2\,l$ swings per day.

18. In the equation for pendulum motion $(ds/dt)^2 = \omega^2 = (2\,g/l)\cos\theta + c$ determine c by the data $\omega = \omega_0$ when $\theta = 0$. Then find what the initial velocity ω_0 must be in order that the pendulum (considered as rigid) rise to the vertical and continue revolving, rather than oscillate.

19. An automobile weighing 1000 lb. turns a corner on the arc of a circle of radius 50 ft. at a speed of 6 mi. per hr. Find the "skid"-producing force. In what ratio would this force be reduced if the speed were halved?

20. A centripetal force of 131 oz. holds a body in a circle of radius 100 ft. with a uniform velocity of one revolution per hr. Find the weight of the body.

21. Formerly railroad curves were built in the form of a circle touched by two straight lines. A later method consisted in using hyperbolic arcs. Explain how this was an improvement.

22. At what angle must the track be raised at a point where its radius of curvature is 400 ft. in order that for trains with a speed of 40 mi. per hr. gravity may exactly balance the pressure on the outer rail due to the normal acceleration?

23. What proportion of its weight does a body lose at the equator because of the earth's rotation? How many times as fast would the earth have to rotate in order that bodies should have no weight at the equator?

24. A hammer thrower whirls a hammer, of weight 16 lb., in a circle of radius 4 ft., and the pull on his arms just before letting go is 180 lb. Find the speed as he lets go, and the length of his throw, assuming the hammer to rise initially at an angle of 45°.

25. A body moves with uniform speed in the parabola $y^2 = 4\,ax$. Determine the normal component of the force. Show that it is greatest at the vertex.

26. Show that if the velocity vector always has the same direction, the motion will be rectilinear.

27. In the elliptic harmonic motion of the present section determine the constants of integration by the following data: when $t = 0$ the body is moving directly upward from the point $(a, 0)$ with a speed kb. Find the equation of the path.

28. On buoyant bodies the upward pressure of the air might be taken into account by assuming for the force $f_x = 0$, $f_y = -mg + mk(h - y)$, where h is a height where the upward pressure practically ceases. Study the motion. Let the body rise from rest.

29. Study the motion of a point repelled from a fixed center by a force proportional to its distance from the center. Show that the path is a hyperbola. Compare with elliptic harmonic motion, and also, as in problem 27 of this section, work out a special case with simple initial conditions.

30. Study the motions obtained from $x = a \cos mt$, $y = b \sin nt$, by giving m and n different integral values. The paths are called Lissajou's curves, and are the result of the composition of two harmonic motions with *different* periods. Show that the whole motion takes place within a rectangle of sides $2\,a$ and $2\,b$. Are the curves all closed or not? Draw a number of them for $m, n = 1, 2, 3, \cdots$.

31. Show that in the motion $x = af(t) + ct + d$, $y = bf(t) + mt + n$, the direction of the force is constant and its magnitude is $\sqrt{a^2 + b^2}f''^2(t)$. Show, conversely, that if the direction of a force is constant, the trajectory must have equations of the above type.

Hint. x'' and y'' are functions of t. As their ratio is constant they may both be regarded as constant multiples of the same function.

32. The enemy's vessel is sailing directly toward us. Our biggest gun can give to its shell a fixed initial speed v_0. Considering the damage done proportional to the speed of the projectile as it strikes the enemy's vessel, show that we get the same effect if we fire at any time after it comes within range. Consider in general terms the effect of the air resistance and the angle at which the shot hits.

33. What is the length of a pendulum beating seconds at a place where $g \doteq 32.2$?

ANALYSIS OF CHAPTER III

1. Subject-matter of dynamics.

2. Rectilinear motion. Definitions of velocity and acceleration.

3. The "complete description" of rectilinear motion.

4. Various special manners of determining rectilinear motion by relations involving velocity and acceleration and the derivation from them of the "complete description." The student should be able to indicate the method of this derivation in any given case.

5. Graphic representations of rectilinear motions.

6. Special rectilinear motions. The student should be able to treat fully the more important of these motions.

7. The "complete description" of plane motion.

8. Definition of velocity and acceleration in plane motion.

9. Some geometric properties of plane motion. The student should be able to derive these, with the possible exception of the derivation of the values of the tangential and normal accelerations. The *results* in these cases he should certainly know.

10. The general differential equations of plane motion.

11. Special plane motions. The student should be able to treat fully the more important of these.

12. Definitions of angular velocity and angular acceleration.

13. The period of a simple pendulum and the determination of g.

14. Planetary motion. The student should know the law of force, the meaning of the integral of areas, and the nature of the path.

CHAPTER IV

WORK AND ENERGY

22. Work. If we move a body against a force, as, for instance, in lifting it against gravity, we do *work* upon the body. We consider first the work done in moving a body in a straight line against a constant force, for instance up an inclined plane. The *amount* of work done against the force is defined to be *minus the product of the distance moved through by the projection of the force upon the line of motion,* or

$$W = - sf_s = - sf \cos(s, f).$$

For instance, if the inclination of an inclined plane be i, $f = mg$ and $\cos(s, f) = \cos[(\pi/2) + i] = -\sin i$, so that if the mass be moved a distance of 2 ft., the work done against gravity is $+ 2\, mg \sin i$. The unit of work in general use is the engineering unit, the *foot pound,* which is the amount of work done in lifting one pound one foot against gravity. The *power* of a machine is the time rate at which it can do work. The unit of power is the *horse power,* or 33,000 ft. lb. per min.

XI. PROBLEMS ON WORK AND POWER

1. Find the horse power necessary to haul a train of 200 T. up a 2% grade at the rate of 20 mi. per hr., frictional resistance being 2 lb. per T.

Solution. The force of gravity on the train is 400,000 lb., and the component of this against the motion is 400,000 $\sin i$ where $\tan i = .02$. By tables we find $\sin i = .0200$, so that we have a retarding force of 8000 lb. due to gravity, and of 400 lb. due to resistance, in all 8400 lb. This is hauled at 20 mi. per hr., that is, at 1760 ft. per min. But each foot means 8400 ft. lb. of work. Thus we have $8400 \times 1760 = 14{,}784{,}000$ ft. lb. per min., or 448 horse power, about.

2. Calculate the work done in raising a weight of 2 T. from the ground to a point a yard above the ground. Suppose inclined planes of inclinations 45°, 30°, 10°, be used, what will be the work done in each case, the final height above the ground being always 1 yd.?

3. A pump discharges water at the rate of 20 cu. ft. per min. from an artesian well 1000 ft. deep. What is its power?

Note. 1 cu. ft. of water weighs $62\frac{1}{4}$ lb.

4. A body of weight m lb. is to be raised from the ground to a height of h ft. by use of an inclined plane making an angle i with the horizontal. If the coefficient of friction is μ, show that the work done will be $hm(1 + \mu \cot i)$ ft. lb.

5. One end of a spring whose other end is clamped is pulled out a distance 18 in., the average force exerted by the spring being 3 poundals. What is the work done?

6. The stroke of a piston whose head has an area of 100 sq. in. is 13 in. long. The average pressure is 80 lb. per sq. in. What is the work done? If the fly wheel makes 10 revolutions per min., what is the estimated horse power?

7. Show that the work done in moving a body from the bottom to the top of a smooth inclined plane is the same as that done in raising the body vertically from the level of the bottom to the top.

23. Force fields; work on curved paths. If a particle moving in space is subject at each point to some definite force which is either constant like gravity or varies from point to point continuously, like the force of a magnet on a piece of iron, the particle is said to move in a *field of force.* Work is done against the field when a body is moved against its forces. Moreover, the path may be curved and the force may vary. In order to extend our definition of work given in the last section, we imagine a polygon inscribed in the curved path and the body to be moved along the sides of this polygon. We shall suppose the force to retain along each side the value it had at the beginning of that side. This will give us an approximation to reality which is closer and closer as the sides of the polygon are decreased in length and increased in number. For each side the work will be $-f_s(s_i)\Delta s_i$, and the definition of the amount of work done in moving the body from the point s_1 of the curve to a point s_2 is accordingly

$$W_{12} = \lim_{\Delta s=0}\left(-\sum f_s(s_i)\Delta s_i\right) = -\int_{s_1}^{s_2} f_s ds.$$

Ex. 1. Show that the work done in raising a particle against gravity from a height h_1 to a height h_2 is independent of the path along which the body is raised, and has therefore the value $(h_2 - h_1)g$.

Ex. 2. Show that the work done against gravity on a solid body by any motion whatever is equal to $(h_2 - h_1) g$ where h_1 and h_2 are the heights at the beginning and the end of the motion, respectively, of the center of mass of the body.

Hint. Assume the whole work is the sum of the work done on the separate parts.

Note that the results of these exercises are valid not only for gravity, but for any force field where the force on the unit mass is constant in magnitude and direction.

XI. PROBLEMS ON WORK AND POWER (*Continued*)

8. Show that the work done in raising a body from the lowest to the highest point of a vertical circle is the same whether a semicircle or the diameter be used as path.

Do this as an independent verification of Ex. 1, not as an application.

9. Find the horse power necessary to haul a train at a speed of 30 mi. per hr., the frictional resistance being equivalent to the weight of 10,000 lb.

10. Let λ be the modulus of elasticity of an elastic string (i.e. the tension required to double its length, it being assumed that increase in the length is proportional to the tension) and x its length. Then the tension is given by $T = (x - l)\lambda / l$, being measured in pounds. Find the work done in stretching the string from a length $x = a$ to a length $x = b$.

11. An automobile ascends a hill, whose inclination is $i = \arctan (1/30)$, a distance of 200 yd. per min. If the automobile weighs 1 T., find the horse power required.

24. Conservative fields; potential energy. If work be done upon a body so as to lift it against gravity, the body, in turn, by means of pulleys, may be made to *do* work in lifting other bodies. If the force field is such that exactly as much work will be done by the body in returning by any path to its original position as was done against it in moving it from this to its final position, the field is called *conservative.* Otherwise it is called *nonconservative*; an example of a nonconservative force is friction. Conservative fields have the property, which the student has proved in the case of gravity, that the *work done in moving a body from a point P_1 to a point P_2 depends only upon the positions of these points and in no sense upon the path between them.*

If a body moves in a conservative field from a point P_1 to a point P_2 against the forces of the field, it acquires a certain *ability to do*

work. This *ability* to do work is called *energy*, and as it depends upon the position of the point it is called *potential* energy, that is, possible energy, ready to manifest itself as soon as the body is allowed to move. It will readily be seen that this energy is so far a relative thing. The excess of energy at P_1 over that at P_0 is defined as equal to the amount of work that can be done by the body in passing from P_1 to P_0, or

$$W_{01} = -\int_{s_0}^{s_1} f_s ds.$$

It is, however, customary to speak of the energy at a point P_1, by which we mean this same difference W_{01}, the point P_0 being considered the standard point of zero energy. There will, in general, be points of negative potential energy, they being of course points such that in moving the body from them to P_0, work must be done against the field. To change the standard point merely means to add a constant to the energy, and this in no wise inconveniences us in our use of the idea. We therefore write

$$W(s) = -\int_{s_0}^{s} f_s ds$$

as the expression for the potential energy.*

Ex. What are the equipotential surfaces of a field of central forces? (See p. 68.)

The projections on the axes of the force at any point of a conservative force field are the negatives of the derivatives of the potential energy with respect to the corresponding coördinates, that is,

$$f_x = -\frac{\partial W}{\partial x}, \qquad f_y = -\frac{\partial W}{\partial y}.$$

For, taking two points on a parallel with the x-axis, with abscissas x_1 and $x_1 + \Delta x_1$, we have

* If all along the path used $f_s = 0$, that is, if the path is at all points perpendicular to the direction of the force, we see that no work is done over the path. All such paths radiating from a point form, in the case of a conservative field, a surface whose normals are the force vectors. Such a surface is called an equipotential surface, for the potential energy is the same at all of its points. Thus if the force of gravity be considered constant, the equipotential surfaces are level planes. From this fact equipotential surfaces for other force fields are frequently called level surfaces.

$$W(x_1+\Delta x_1)-W(x_1)=-\int_{x_0}^{x_1+\Delta x_1} f_x dx+\int_{x_0}^{x_1} f_x dx=-\int_{x_1}^{x_1+\Delta x} f_x dx,$$

or, applying the law of the mean for integrals,

$$\Delta W=-f_x(x_1+\theta\Delta x_1)\int_{x_1}^{x_1+\Delta x_1} dx=-f_x(x_1+\theta\Delta x_1)\,\Delta x_1,$$

where $0<\theta<1$. Dividing by Δx_1 and taking the limit,

$$\frac{\partial W}{\partial x}=-f_x, \quad \text{and similarly} \quad \frac{\partial W}{\partial y}=-f_y.$$

From this we derive an interesting corollary: *A particle is in equilibrium when its potential energy is at a maximum or minimum.*

The converse, that if the body is in equilibrium, the potential energy is at a maximum or minimum, is *not* always true, though frequently so stated. For example, if $W=x^2-y^2$, $f_x=f_y=0$ at the origin, but the surface $z=x^2-y^2$ is saddle shaped here, and has neither a maximum nor a minimum.

XII. PROBLEMS ON POTENTIAL ENERGY

1. The potential energy of a solid body or a system is a maximum or minimum when the height of the center of mass is a maximum or minimum. Two rods of length l hinged together like an inverted V lie over a cylinder of radius a. Let $2\,\theta$ be the angle between them in a position of equilibrium. Show that θ satisfies the equation $2\,a\cos\theta=l\sin^3\theta$.

2. A right cylinder with elliptic cross section is so weighted that its center of mass is in a plane halfway between its parallel bases and one quarter of the way along its maximum diameter. Find the possible positions of equilibrium of the cylinder lying horizontally on a smooth plane, and show that there are four or two according or not as the eccentricity is greater than $1/\sqrt{2}$.

Hint. This requires finding the distance of the point $(a/2, 0)$ from the point (x, y) on the ellipse $x=a\cos\theta$, $y=b\sin\theta$ where $b^2=a^2(1-e^2)$.

25. Kinetic energy. A body may have energy because of motion. Thus if a body is thrown upward against gravity it continually gains potential energy until it reaches its highest point. The energy which its motion gives it is called its *kinetic energy,* and is measured by *the amount of work the body does against the forces of the field before coming to rest.*

For this we find (confining ourselves to the plane)

$$k = -\int_{s}^{s_0} f_s ds,$$

where s_0 determines the position of rest. But by IV of § 5 this is

$$-\int_{s}^{s_0} [f_x \cos(x, s) + f_y \cos(y, s)]\, ds,$$

or, since $\cos(x, s) = \dfrac{dx}{ds}, \qquad \cos(y, s) = \dfrac{dy}{ds},$

$$k = -\int_{x, y}^{x_0, y_0} (f_x dx + f_y dy) = -\int_{t}^{t_0} \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \right) dt;$$

and since $f_x = m \dfrac{dv_x}{dt}, \qquad f_y = m \dfrac{dv_y}{dt},$

$$k = -m \int_{t}^{t_0} \left(\frac{dv_x}{dt} v_x + \frac{dv_y}{dt} v_y \right) dt$$

$$= -m \int_{t}^{t_0} \frac{d}{dt} \left(\frac{1}{2} v^2 \right) dt = -\frac{m}{2} v_0^2 + \frac{m}{2} v^2,$$

or, since the position characterized by t_0 is one of rest, $v_0 = 0$, and we have

$$k = \tfrac{1}{2} mv^2.$$

26. Conservation of energy. Suppose a body move in a force field from a point P_1 to a point P_2; the gain in kinetic energy is $\frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2$. But this is equal to the work done by the forces of the field, or

$$\int_{s_1}^{s_2} f_s ds = -W(s_2) + W(s_1),$$

so that, equating and transposing,

$$\tfrac{1}{2} mv_2^2 + W(s_2) = \tfrac{1}{2} mv_1^2 + W(s_1),$$

that is, *the sum of the kinetic and potential energies of a body is always the same during motion in a conservative force field.* This result is known as the principle of the conservation of energy. As the object of a perpetual-motion machine is to do work (at least against the friction of its own parts) perpetually, always returning to some standard position with the same kinetic energy, we see that *a perpetual-motion machine is impossible, provided the forces on which it depends are conservative.*

Ex. Suppose a body moves on a curved path under the influence of gravity, like a head upon a smooth wire. By the principle that the energy is constant, show that the speed of the body is the same at all points where it has the same heights (cf. problem 10 of § 21).

XIII. PROBLEMS ON ENERGY

[See note under Suggestions and Answers.]

1. A hammer weighing 1 lb. strikes a nail with a velocity 8 ft. per sec., driving it a quarter of an inch into a plank. Find the average force exerted upon the nail during its motion.

Solution. The kinetic energy of the hammer is transferred to the nail, except a small amount which is dissipated in heat, which we neglect here. Thus $\frac{1}{2}mv^2 = \frac{1}{2}(1/32.2)8^2 = .99$ ft. lb. work is done against resistance by the nail. If the force is constant, we have for the work $f \cdot s = f \cdot \frac{1}{4} \cdot \frac{1}{12}$, and as this is equal to the kinetic energy, we have $f = 48 \cdot .99 = 48.3$ lb.

2. How much work is accumulated in a body weighing 300 lb. and moving 64 ft. per sec.?

3. A fly wheel 12 ft. in diameter, whose rim weighs 12 T., makes 50 revolutions per min. What is its kinetic energy?

4. An automobile running 30 mi. per hr. comes to the foot of a hill. To what height will it ascend without power, friction and other resistance being neglected?

5. Compare the kinetic energy of a mass of 20 lb. falling from rest at the end of the fifth second with the energy at the end of the sixth second.

6. A mason's helper throws bricks up to him through a vertical distance of 14 ft., so that when the mason catches them they have a vertical velocity of 6 ft. per sec. By what proportion would the helper reduce his work by throwing the brick so as to reach the mason with no speed remaining?

7. If a pendulum hanging at rest is given an initial velocity v_0, how high will it rise?

8. A gun carriage of 2 T. recoils horizontally with a velocity of 12 ft. per sec. Find the constant force which will take up the recoil within a distance of 2 ft.

9. The 500-lb. hammer of a pile driver falls 10 ft. on to the head of a pile, which is forced thereby an inch into the ground. Find the average force exerted by the hammer.

10. A river has a cross section of area 16,000 sq. ft. and flows with a mean speed of 4 mi. per hr. Find the horse power that would be developed if all the energy of the river could be utilized.

Note. The weight of 1 cu. ft. of water is $62\frac{1}{4}$ lb.

11. A particle slides from the highest point on the outer surface of a smooth sphere of radius a. Find the point where the particle leaves the sphere.

ANALYSIS OF CHAPTER IV

1. Definitions and units of work and power.
2. Definition of work for a curved path and varying force.
3. Definitions of conservative fields and potential energy.
4. The force components given by the negative of the derivatives of the potential energy; equilibrium in terms of potential energy.
5. Definition of and expression for kinetic energy.
6. The principle of the conservation of energy.

CHAPTER V

MECHANICS OF RIGID BODIES

27. Instantaneous motion of a rigid body. When a body moves from one position to another, and we consider only the positions, and leave out of account the way in which the body moved from one to the other, we speak of the *displacement* of the body. The displacement of any point of the body may be represented by a vector joining its initial and final positions. We next consider two special *motions* of a body, in which the intermediate positions are considered: namely, *translation,* in which all points move in congruent curves, for which we may usually take straight lines, and all lines in the body remain parallel to themselves; and *rotation,* in which there is a line, or axis, in the body, all of whose points are fixed during the motion. The following theorem is important:

Any displacement of a rigid body may be brought about by a translation and a rotation. To see this we remark that the motion of a body may be studied by fixing a set of coördinate axes in the body and considering their various positions relative to a set of axes fixed in space. Let us denote by O_1X_1, O_1Y_1, O_1Z_1 the initial position of the axes fixed in the body, and by O_2X_2, O_2Y_2, O_2Z_2 their final position. The translation desired is, then, the one which carries O_1 over into O_2. Let us suppose it carries O_1X_1, O_1Y_1, O_1Z_1 into the parallel set O_2X_1', O_2Y_1', O_2Z_1'. We have now to show that a rotation may be found which carries O_2X_1', O_2Y_1', O_2Z_1' into O_2X_2, O_2Y_2, O_2Z_2 respectively. Let X_1', Y_1', Z_1', X_2, Y_2, Z_2, denote the points where the corresponding axes pierce a unit sphere about the point O_2 (see Fig. 31). Then we need merely show that we can find a diameter such that, if the sphere be rotated about it through the proper angle, X_1' will pass into X_2 and Y_1' into Y_2, for then Z_1' will

also become Z_2 because each Z-point is a quadrant's distance from the corresponding X- and Y-points and on the same side of the XY-plane in both cases. Now in this rotation which we seek, all points of the sphere move on parallel circles with the required diameter as axis. Hence they remain unchanged in their distance from this pole; the pole is equidistant from X_1' and X_2, and is therefore in the plane which perpendicularly bisects the line join-

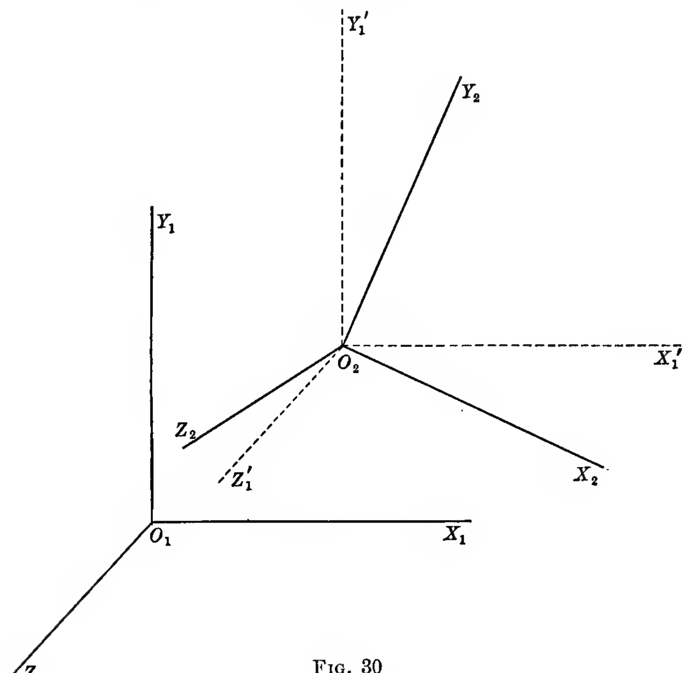


Fig. 30

ing them. Similarly, it lies in the plane which perpendicularly bisects $Y_1'Y_2$. The axis is thus determined, unless the planes coincide. It is left to the student to consider the meaning of this case, and further to prove that an *angle* can always be found which will serve to bring *both* X_1' into X_2 and Y_1' into Y_2 at the same time.

It is evident that this resolution of a displacement into translation and rotation can be effected in many ways, for instead of

choosing the particular point O_1 as origin of our axis in the body we might have chosen any other point. It is important to notice that the translation and rotation are by no means the actual motion of the body. They simply give a way in which it may be brought from its initial to its final position. The real value of this resolution consists in its application as follows: In a short interval of time, Δt, the body suffers a small displacement, which may be resolved as indicated. The displacement of any given one of its points is thus also resolved into one due to the translation and one due to the rotation. The smaller Δt the more accurately do these two displacements actually represent the real motion of the point. Considering the displacements as vectors, and dividing by Δt and taking the limit, we see that *at any instant* the velocity of any given point may be considered as decomposed into *a velocity of translation* and *a velocity of rotation;* or, as we frequently express it, the instantaneous motion consists in an *instantaneous translation* and an *instantaneous rotation.* The important thing, however, is that the velocity of translation and the angular velocity of the rotation are the same for all points of the rigid body. Thus instead of having an infinite number of velocities to deal with corresponding to the infinitely many points, as we should have in a nonrigid body, we have merely two. Similar results hold for the acceleration. The actual finite motion of the body is generated by a translation with a velocity that, in general, is continually changing, and by a rotation with varying velocity about a varying axis.

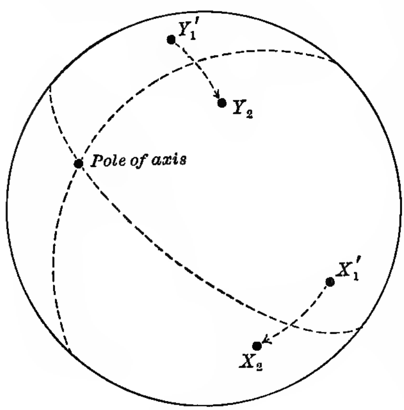


FIG. 31

The foregoing remarks make clear the importance of studying two particular cases of motion of a rigid body, namely pure translation

and pure rotation. Before doing so, however, two general remarks should be made.

The first concerns the internal forces and reactions of the body. We shall make the assumption embodied in Newton's third law of motion, that the forces exerted by two particles upon each other lie in the line joining them, and are equal in magnitude and opposite in direction. This assumption will enable us to leave out of consideration entirely the effect of internal forces in the study of rigid bodies, as we shall see later on.

The second concerns the energy of a system of a number of particles. If two weights are raised against gravity, the amount of work to be obtained by letting them fall is the sum of the amounts to be obtained from each singly. Similarly, for any number of particles. We are thus led to the more general statement: *The potential and kinetic energies of any system of masses are the sums of the potential and kinetic energies, respectively, of the individual masses,* although we shall not take the space to consider the complete establishment of this principle.

It would be a mistake, however, to assume that if a velocity is resolved into two component velocities, the kinetic energy is the sum of the energies due to the component velocities, as is shown by the simple case in which we consider rest as the resultant of two equal but oppositely directed velocities. If the speed is v, we should have for the sum of the kinetic energies $\frac{1}{2}mv^2 + \frac{1}{2}mv^2$, whereas the total energy is obviously 0. We shall see later what statement can be made in this respect concerning a rigid body (§ 31).

28. Pure translation. *The work done upon a rigid body during a translation is the same as if the whole mass were concentrated at the center of mass.* To see this we imagine the body split up, after the fashion of the Integral Calculus, into a set of masses which are approximately particles. Then all the elements of mass Δm move along congruent curves with the same acceleration, and the same is true of the center of mass. If a_s is the projection of this common acceleration upon the common direction of the paths, the projection of the force acting upon the whole mass M as if concentrated at the center of mass upon the direction of its path is

$a_s M$, and hence the work done on this hypothesis is

$$-\int_{s_1}^{s_2} a_s M ds.$$

As a_s is also the acceleration of the mass Δm, the total force acting upon it is $a_s \Delta m$, though it must not be inferred that this force comes alone from the external field, for the internal stresses of the body will contribute a part. These forces contribute no work, however, because they occur always in pairs, and the distances between their points of application remain unchanged. Hence the work for the mass Δm is

$$W_i = -\int_{s_1}^{s_2} a_s \Delta m_i ds;$$

and summing and taking the limit, we have for the total work

$$\begin{aligned} W &= \lim \sum_i \left(-\int_{s_1}^{s_2} a_s \Delta m_i ds \right) \\ &= -\int_{s_1}^{s_2} a_s \lim \sum_i \Delta M_i ds = -\int_{s_1}^{s_2} a_s M ds, \end{aligned}$$

which agrees with the above result. In replacing the summations by the integrals which are their limits, we eliminate any error arising from the fact that the masses Δm_i are not particles, and thus the theorem is established.

Ex. Prove similarly that the kinetic energy also of a rigid body whose motion is a translation is the same as if the mass were concentrated at the center of mass.

29. Pure rotation. We ask first for the work done by a rotation, in which we consider one force F as acting, applied at a point (x, y, z). Let us take the z-axis along the axis of rotation. The work done against the force is, by definition,

$$W_{12} = -\int_{s_1}^{s_2} f_s ds.$$

Let r denote the length of the perpendicular upon the axis from (x, y, z), $(r = \sqrt{x^2 + y^2})$, and let θ be the angle this line makes with the xz-plane. Then if s is measured from the point where the circular path of (x, y, z) pierces the xz-plane, $s = r\theta$ and $ds = r d\theta$;

then, as $\cos(x, s) = \cos[\theta + (\pi/2)] = -\sin\theta = -y/r$, $\cos(y, s) = \cos\theta = x/r$, and $\cos(z, s) = 0$, $f_s = f_x \cos(x, s) + f_y \cos(y, s) + f_z \cos(z, s) = (xf_y - yf_x)/r$, and we have

$$W = -\int_{\theta_1}^{\theta_2} (xf_y - yf_x)\, d\theta.$$

The quantity $xf_y - yf_x$ is the *moment* of the force about the z-axis (see § 11, II, also Ex. 2). Thus *the work done against a force by a rotation is the negative of the integral of the moment of the force over the angle rotated through.* If we have a number of forces applied at different points,

$$W_{12} = -\int_{\theta_1}^{\theta_2} \sum (x_i f_{iy} - y_i f_{ix})\, d\theta.$$

Here again the interior forces may be left out of consideration, for their moments are equal and opposite in pairs.

The kinetic energy of the mass m rotating about the z-axis is $\frac{1}{2} mv^2$. But as $s = r\theta$, $v = ds/dt = r(d\theta/dt) = r\omega$, this expression becomes $\frac{1}{2} mr^2\omega^2$, and for a number of masses

$$K = \sum \tfrac{1}{2} m_i r_i^2 \omega^2 = \tfrac{1}{2}\left(\sum m_i r_i^2\right)\omega^2.$$

If our masses form a rigid continuous body, we find, by the process of the Integral Calculus,

$$\begin{aligned} K &= \lim \sum \tfrac{1}{2}\, \delta(x_i, y_i, z_i)\, r_i^2 \Delta x_i \Delta y_i \Delta z_i\, \omega^2 \\ &= \frac{1}{2}\left[\iiint \delta(x, y, z)\, r^2 dxdydz\right]\omega^2. \end{aligned}$$

The quantities

$$mr^2, \quad \sum m_i r_i^2, \quad \text{and} \quad \iiint \delta(x, y, z)\, r^2 dxdydz,$$

which multiply $\frac{1}{2}\omega^2$ in the above expressions for the kinetic energy, are called the *moments of inertia* of the particle, system of particles, or the continuous body respectively. It will be noticed that if we consider angular velocity in rotation as corresponding to speed in translation, they occupy exactly the same place with respect to the energy of rotation that the mass does with respect to the energy of translation; they are the coefficients of half the angular speed; they measure the inertia or resistance of the body

to turning forces. This fact makes them of great importance both in the theory of rotating bodies and also in other problems of engineering, where similar expressions find application to the theory of strength of materials. We shall therefore devote a separate paragraph to them.

30. Moments and products of inertia. Radius of gyration. In the integral

$$I=\iiint \delta(x, y, z)\, r^2 dxdydz$$

the limits are determined just as in the volume problems of the Integral Calculus, by the boundaries of the body considered. The expression holds for any axis, where r denotes the distance of the point x, y, z from that axis. In particular

$$A=\iiint \delta(x, y, z)(y^2+z^2)\, dxdydz,$$

$$B=\iiint \delta(x, y, z)(z^2+x^2)\, dxdydz,$$

$$C=\iiint \delta(x, y, z)(x^2+y^2)\, dxdydz,$$

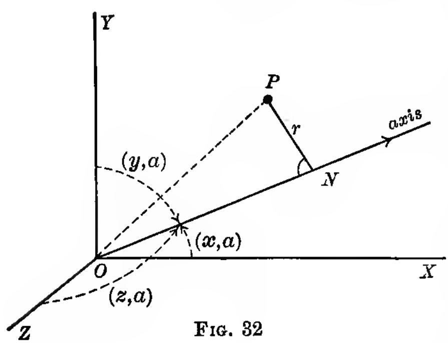


FIG. 32

are the moments of inertia about the three coördinate axes. If we have any other axis a, the distance r of the point (x, y, z) from this axis is one side of a right triangle, of which the hypotenuse OP is the distance $\sqrt{x^2+y^2+z^2}$ of (x, y, z) from the origin, and one side ON is the projection* $x\cos(x, a)+y\cos(y, a)+z\cos(z, a)$ of this distance upon the axis a. We have, therefore,

* This is merely an extension to space of IV of § 5. OP is a vector with projections x, y, z on the axes.

$$
\begin{aligned}
r^2 &= OP^2 - ON^2 \\
&= [x^2+y^2+z^2] - [x\cos(x,a) + y\cos(y,a) + z\cos(z,a)]^2 \\
&= x^2[1-\cos^2(x,a)] + y^2[1-\cos^2(y,a)] + z^2[1-\cos^2(z,a)] \\
&\qquad - 2\,yz\cos(y,a)\cos(z,a) - 2\,zx\cos(z,a)\cos(x,a) \\
&\qquad - 2\,xy\cos(x,a)\cos(y,a);
\end{aligned}
$$

or, remembering that $\cos^2(x,a)+\cos^2(y,a)+\cos^2(z,a)=1$,

$$
\begin{aligned}
r^2 = \cos^2(x,a)[y^2+z^2] &+ \cos^2(y,a)[z^2+x^2] + \cos^2(z,a)[x^2+y^2] \\
&- 2\cos(y,a)\cos(z,a) - 2\cos(z,a)\cos(x,a)\,zx \\
&- 2\cos(x,a)\cos(y,a)\,yx.
\end{aligned}
$$

Multiplying by $\delta(x, y, z)$ and integrating, and remembering that $\cos(x, a)$, $\cos(y, a)$, $\cos(z, a)$, being the cosines of the angles between the axis a and the coördinate axes are therefore constant, we find

$$
\begin{aligned}
I &= \iiint \delta(x,y,z)\, r^2\, dxdydz \\
&= A\cos^2(x,a) + B\cos^2(y,a) + C\cos^2(z,a) - 2\,D\cos(y,a)\cos(z,a) \\
&\quad - 2\,E\cos(z,a)\cos(x,a) - 2\,F\cos(x,a)\cos(y,a), \qquad (1)
\end{aligned}
$$

where
$$
\begin{aligned}
D &= \iiint \delta(x,y,z)\, yz\, dxdydz, \\
E &= \iiint \delta(x,y,z)\, zx\, dxdydz, \\
F &= \iiint \delta(x,y,z)\, xy\, dxdydz,
\end{aligned}
$$

are the so-called *products of inertia.* Thus having calculated the six *coefficients of inertia*, A, B, C, and D, E, F, we can find from them the moment of inertia about any axis through the origin by means of the formula (1) above. The coefficients are usually calculated for coördinate axes with origin at the center of mass. The following theorem then enables us to find the moment of inertia about any axis, even if not through the center of mass, without further integrations, unless we need one to find the mass: *The moment of inertia about any given axis is equal to the moment about a parallel*

axis through the center of mass increased by the moment of inertia about the given axis of the body considered concentrated at its center of mass.

In symbols, if I' denote the required moment of inertia and I the moment about a parallel through the center of mass, and a the distance between the parallel axes,

$$I' = I + Ma^2. \tag{2}$$

To prove this, we take our coördinate axes through the center of mass, so that the z-axis is parallel to the given axis, and so that

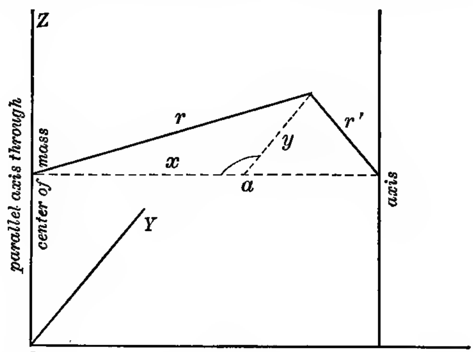


FIG. 33

the xz-plane passes through it. The equations of the given axis will then be $x = a$, $y = 0$. If, then, r' denote the distance of the point (x, y, z) from the given axis, we have

$$r'^2 = (x - a)^2 + y^2 = x^2 + y^2 - 2\,ax + a^2 = r^2 + a^2 - 2\,ax.$$

Hence

$$I' = \iiint \delta(x, y, z)\, r'^2 dxdydz = \iiint \delta(x, y, z)\, r^2 dxdydz$$
$$+ a^2 \iiint \delta(x, y, z)\, dxdydz - 2\,a \iiint \delta(x, y, z)\, x\, dxdydz.$$

The first integral on the right is I, the second M, and the third is $\bar{x}M$ (see § 13). But as the origin was taken at the center of mass, $\bar{x} = 0$, so that this term drops out and equation (2) is thus established.

If the whole body were concentrated on an axis, the moment of inertia about that axis would vanish. We ask, *How far from the axis should the mass of the body be concentrated in order that its moment of inertia with respect to that axis be the same as when the body has its given form?* The distance sought is called the *radius of gyration* and is denoted by R. From its definition

$$MR^2 = I. \quad \text{Hence} \quad R = \sqrt{\frac{I}{M}}\cdot$$

XIV. PROBLEMS ON MOMENTS AND PRODUCTS OF INERTIA AND RADII OF GYRATION

Unless otherwise specified, take the density (x, y, z) equal to 1. Choose the coördinate axes as symmetrically as possible. Calculate A, B, C, D, E, F and the radii of gyration about the coördinate axes. Straight wires should be laid along the x-axis, and plates in the xy-plane.

The student should observe that sometimes we can tell in advance of a definite integral that it vanishes, namely, in cases where the field of integration is symmetric with respect to a point, line, or plane, and the integrand has values which are equal in absolute value but opposite in sign at pairs of symmetric points. Thus

$$\int_{-1}^{+1} x dx = 0, \quad \int_{-a}^{+a} x^5 dx = 0, \quad \int_0^{\pi} \cos x dx = 0,$$

$$\int_{-a}^{+a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} xy dy dx = 0, \quad \int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c} \sin\pi\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right) dz dy dx = 0.$$

The truth of this statement is obvious when we recall the fact that the integral is the limit of a sum.

Two bodies with the same coefficients of inertia are sometimes called *dynamically equivalent.*

1. Calculate the coefficients of inertia of an equilateral triangular plate of side a, and of unit surface density; also the moment of inertia about a side, about an axis perpendicular to the plate and through one corner, and finally about an axis through one corner and making an angle of 45° with the plate and lying in the plane that perpendicularly bisects the opposite edge of the plate. Give also the radii of gyration.

Solution. The center of mass is at the intersection of the medians, which we take for origin of coördinates, taking one median along the y-axis. We find the equations of the sides to be $3x + \sqrt{3}y - a = 0$, $3x - \sqrt{3}y + a = 0$, and $2\sqrt{3}y + a = 0$. Then

$$A = \iint (y^2 + z^2)\, dm,$$

and since we have a plate, $z = 0$, and with unit density $dm = dxdy$, so that we have

$$A = \int\int y^2\,dxdy = \int_{-\frac{a}{2\sqrt{3}}}^{\frac{a}{\sqrt{3}}} \int_{-\frac{a-\sqrt{3}y}{3}}^{\frac{a-\sqrt{3}y}{3}} y^2\,dxdy = \int_{-\frac{a}{2\sqrt{3}}}^{\frac{a}{\sqrt{3}}} y^2\,\tfrac{2}{3}(a-\sqrt{3}\,y)\,dy$$

$$= \frac{2}{3}\left[\frac{ay^3}{3} - \frac{\sqrt{3}\,y^4}{4}\right]_{-\frac{a}{2\sqrt{3}}}^{\frac{a}{\sqrt{3}}} = \frac{a^4}{32\sqrt{3}}.$$

Similarly,

$$B = \int\int x^2\,dxdy = \int_{-\frac{a}{2\sqrt{3}}}^{\frac{a}{\sqrt{3}}} \int_{-\frac{a-\sqrt{3}y}{3}}^{\frac{a-\sqrt{3}y}{3}} x^2\,dxdy = \frac{2}{3}\int_{-\frac{a}{2\sqrt{3}}}^{\frac{a}{\sqrt{3}}} \left(\frac{a-\sqrt{3}\,y}{3}\right)^3 dy$$

$$= -\frac{2}{\sqrt{3}}\cdot\frac{1}{4}\left(\frac{a-\sqrt{3}\,y}{3}\right)^4 \Bigg|_{-\frac{a}{2\sqrt{3}}}^{\frac{a}{\sqrt{3}}} = \frac{2}{\sqrt{3}}\cdot\frac{1}{4}\left(\frac{a+\frac{a}{2}}{3}\right)^4 = \frac{a^4}{32\sqrt{3}}.$$

$$C = \int\int (x^2+y^2)\,dxdy = A + B = \frac{a^4}{16\sqrt{3}}.$$

Moreover, $D = \int\int yzdxdy = 0$, and $E = \int\int zxdxdy = 0$ since $z = 0$, and $F = \int\int xydxdy = 0$ because at points of the plate that are symmetric with respect to the y-axis, (x, y) has numerical values but opposite signs.

To find the moment of inertia about a side, we select the side that is parallel to the x-axis and employ equation (2). The x-axis passes through the center of mass and the moment about it is A. Its distance from the side is $a/2\sqrt{3}$. Hence for a side

$$I = A + \left(\frac{a}{2\sqrt{3}}\right)^2\cdot\left(\frac{\sqrt{3}}{4}a^2\right),$$

the mass being the same as the area. This gives

$$I = \frac{a^4}{32\sqrt{3}} + \frac{\sqrt{3}\,a^4}{16\cdot 3} = \frac{\sqrt{3}\,a^4}{32}.$$

The same method gives us the moment about a perpendicular through a corner. Take the corner on the y-axis. We have, then,

$$I = C + \left(\frac{a}{\sqrt{3}}\right)^2\left(\frac{\sqrt{3}}{4}a^2\right) = \frac{a^4}{16\sqrt{3}} + \frac{a^4}{4\sqrt{3}} = \frac{5\,a^4}{16\sqrt{3}}.$$

Finally, to find the moment about the axis through a corner, and inclined at an angle of 45° to the plate, we find first by equation (1) the moment about the parallel axis through the center of mass. We have for this axis the angles $(x, a) = 90°$, $(y, a) = 135°$, $(z, a) = 45°$, and hence $\cos(x, a) = 0$, $\cos(y, a) = -1/\sqrt{2}$, $\cos(z, a) = 1/\sqrt{2}$. Whence for this axis, by equation (1),

$$I = \frac{B}{2} + \frac{C}{2} = \frac{\sqrt{3}\,a^4}{64},$$

and it remains to find the moment about a parallel axis, whose distance, measured along the plate, is $a/\sqrt{3}$, but measured along a common perpendicular is $a/\sqrt{3}\cos 45° = a/\sqrt{6}$. Hence for the required axis through the corner

$$I = \frac{\sqrt{3}\,a^4}{64} + \left(\frac{a}{\sqrt{6}}\right)^2\left(\frac{\sqrt{3}}{4}\,a^2\right) = \frac{11\,a^4}{64\sqrt{3}}.$$

The radii of gyration are all found by the formula $R = \sqrt{I/M}$, and are respectively: about OX, $a/2\sqrt{6}$; about OY, $a/2\sqrt{6}$; about OZ, $a/2\sqrt{3}$; about a side, $a/2\sqrt{2}$; about a perpendicular through a corner, $5\,a/2\sqrt{3}$; about the slanting axis, $\sqrt{11}\,a/4\sqrt{3}$.

2. Show that for straight wires $A = 0$, $B = C$, $D = E = F = 0$.

3. Show that for plates $C = A + B$, $D = E = 0$.

4. Show that the moment of inertia of a body about an axis through the center of mass is less than about any parallel axis.

5. Calculate B for a straight homogeneous wire of length l. Calculate its moment of inertia about an axis through its center of mass and inclined at 45° to it. Calculate independently, and by equation (2) of the present paragraph, the moment of inertia about a perpendicular axis through one end.

6. Calculate the coefficients of inertia for a uniform square plate of side l. Find the moments of inertia about a diagonal and about a perpendicular to the plate through one corner; also about a perpendicular through the midpoint of one side.

7. Calculate the coefficients for a homogeneous cube of side l. Calculate the moments of inertia about an edge and about a diagonal. (For the diagonal $\cos(x, a) = \cos(y, a) = \cos(z, a)$, and as $\cos^2(x, a) + \cos^2(y, a) + \cos^2(z, a) = 1$, each can easily be found.) Show that for all axes through the center of mass I is the same.

8. Calculate the coefficients of inertia of a homogeneous circular plate of radius r. What must be the radius of the plate to have the same coefficients as the square of problem 5? The square is dynamically equivalent to this disk.

9. Calculate the coefficients of inertia of a homogeneous sphere of radius r. What must r be in order that the coefficients agree with those of the cube in problem 6? Calculate the moment about a line through the center with direction angles (x, a), (y, a), (z, a). The cube is dynamically equivalent to this sphere.

10. Calculate the coefficients for a wire ring of radius r.

11. Calculate C for an arc corresponding to an angle $2\,a$ of the above wire ring, the origin being at the center of mass.

12. Calculate the coefficients of inertia of the elliptic plate $x^2/a^2 + y^2/b^2 = 1$. Calculate its moments of inertia about axes through its focus parallel with its minor axis and perpendicular to its plane.

13.* Calculate the coefficients of inertia of the homogeneous ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

14. Show that if a homogeneous body is symmetric with respect to the xy-plane, $D = 0$. What further similar statements can you make regarding bodies with one or more planes of symmetry?

15.* Find the moment of inertia of the cardioid plate bounded by $\rho = a(1 - \cos\theta)$ about an axis through the origin and perpendicular to its plane; also about the axis of symmetry in the plane.

16. Determine the coefficients of inertia of a cuboid of dimensions a, b, and c; also moments of inertia about edges and about a diagonal.

17. Find the moments of inertia of the above cuboid about the diagonals of the rectangles forming its faces.

18.* Find the moment of inertia of the elliptic plate $l/r = 1 - e\cos\theta$ about a perpendicular through the focus (which is the origin as the equation is written). Compare the result with problem 12, noting that $e^2 = (a^2 - b^2)/a^2$ and $l = b^2/a$.

19. Find the moment of inertia about its axis of a right circular cone of altitude a and radius of base r.

20. Find the coefficients of inertia of a right circular cylinder of height a and radius of base r.

21. Calculate the coefficients of inertia of a flat angle iron as indicated in Fig. 17, p. 39, the origin being the center of mass.

22. Do the same for the plates of Figs. 14 and 15, taking simple numerical values for the constants.

23. What is the effect on the moments of inertia and the radius of gyration of a homogeneous body if the density is multiplied by a constant k, that is, increased in the ratio $1:k$?

24. In problem 5 calculate the moment of inertia of the wire about an axis through its mid-point and making an angle θ with it. How does the moment of inertia change as θ varies from 0 to $\pi/2$? Can you explain this from mechanical considerations?

25. Given a circular plate of radius r from which have been removed four circular pieces of radius r' with centers at the mid-points of four equally spaced radii of the plate. What is the value of r' if the radius of gyration of the plate about an axis perpendicular to its plane through its center is $\frac{3}{4}$ the radius of the plate?

26. Suppose a body expand in such a way that it always remains similar to itself in form and that its mass is constant in amount. How are its coefficients of inertia and its radii of gyration affected?

31. Work and energy in the case of a rigid body. Let us consider first a system of particles m_i with coördinates (x_i, y_i, z_i) with

* The reckonings in the starred problems are rather protracted. Integration tables should be used, e.g. Peirce (formulas 308 and 300 for problem 18).

respect to a fixed set of axes. As we are about to study how the work and energy of the body depends upon the work and energy due to translation and rotation separately, we shall introduce also a system of axes moving parallel with the old, with origin always at the center of mass $\bar{x}$, $\bar{y}$, $\bar{z}$. If (x_i', y_i', z_i') are the coördinates of a point with respect to these axes, we have, by Analytic Geometry,

$$\left.\begin{aligned} x_i &= \bar{x} + x_i' \\ y_i &= \bar{y} + y_i' \\ z_i &= \bar{z} + z_i' \end{aligned}\right\}. \tag{1}$$

The work done against the force field in moving m_i will be

$$\begin{aligned} W_i &= -\int_{s_{i_2}}^{s_{i_1}} f_{si} ds_i \\ &= -\int_{s_{i_1}}^{s_{i_2}} [f_{xi}\cos(x_i, s_i) + f_{yi}\cos(y_i, s_i) + f_{zi}\cos(z_i, s_i)] ds_i, \end{aligned}$$

or, as $\cos(x_i, s_i) = \dfrac{dx_i}{ds_i}$, $\cos(y_i, s_i) = \dfrac{dy_i}{ds_i}$, $\cos(z_i, s_i) = \dfrac{dz_i}{ds_i}$,

$$\begin{aligned} W_i &= -\int_{s_{i_1}}^{s_{i_2}} \left[f_{xi}\frac{dx_i}{ds_i} + f_{yi}\frac{dy_i}{ds_i} + f_{zi}\frac{dz_i}{ds_i} \right] ds_i \\ &= -\int_{t_1}^{t_2} \left[f_{xi}\frac{dx_i}{dt} + f_{yi}\frac{dy_i}{dt} + f_{zi}\frac{dz_i}{dt} \right] dt, \end{aligned}$$

or $$W_i = -\int_{t_1}^{t_2} [f_{xi}v_{xi} + f_{yi}v_{yi} + f_{zi}v_{zi}]\, dt. \tag{2}$$

Now, differentiating (1) with respect to t, we have

$$\left.\begin{aligned} v_{xi} &= v_{\bar{x}} + v_{xi}' \\ v_{yi} &= v_{\bar{y}} + v_{yi}' \\ v_{zi} &= v_{\bar{z}} + v_{zi}' \end{aligned}\right\}, \tag{3}$$

and using these values, we have

$$W_i = -\int_{t_1}^{t_2} [f_{xi}v_{\bar{x}} + f_{yi}v_{\bar{y}} + f_{zi}v_{\bar{z}}]\, dt - \int_{t_1}^{t_2} [f_{xi}v_{xi}' + f_{yi}v_{yi}' + f_{zi}v_{zi}']\, dt.$$

If now we have a number of particles, the work done against them all is the sum of the work done against them separately (see p. 89). In summing we should recall that $v_{\bar{x}}$, $v_{\bar{y}}$, $v_{\bar{z}}$, are the same for every

term in the sum, so that we simply add their coefficients. The sums

$$X = \sum_i f_{xi}, \quad Y = \sum_i f_{yi}, \quad Z = \sum_i f_{zi},$$

are the projections on the axes of the total forces acting on the system. The result is

$$W = -\int_{t_1}^{t_2} [Xv_x + Yv_y + Zv_z]\, dt - \sum \int_{t_1}^{t_2} [f_{xi}v'_{xi} + f_{yi}v'_{yi} + f_{zi}v'_{zi}]\, dt, \quad (4)$$

in which, by comparing with (2), we see that the second term is the total work computed as if the center of mass were at rest, for v'_{xi}, v'_{yi}, v'_{zi}, are the velocities *relative* to the center of mass.

The considerations apply in particular to rigid bodies. In either case the theorem is true: *The work done in moving a rigid body is equal to the work done in moving the center of mass computed as if all the forces were applied there, plus the whole work done on the body computed as if its center of mass were at rest.*

In a conservative force field, inasmuch as the work done is for each particle the same, no matter what path is taken, so for a rigid body the work depends only upon the initial and final positions. As the change may be brought about by a translation and a rotation about the center of mass, we may say that *in moving a rigid body against a conservative force field, the work done is the sum of the work done in translating its center of mass, the forces being regarded as concentrated there, plus the work done in rotating the body about its center of mass.*

Similar results hold for the kinetic energy. We have for the particle m_i,

$$\begin{aligned} k_i &= \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2} m_i [v_{xi}^2 + v_{yi}^2 + v_{zi}^2] \\ &= \tfrac{1}{2} m_i [(v_{\bar{x}} + v'_{xi})^2 + (v_{\bar{y}} + v'_{yi})^2 + (v_{\bar{z}} + v'_{zi})^2] \\ &= \tfrac{1}{2} m_i \bar{v}^2 + \tfrac{1}{2} m_i v_i'^2 + m_i [v_{\bar{x}} v'_{xi} + v_{\bar{y}} v'_{yi} + v_{\bar{z}} v'_{zi}]. \end{aligned}$$

When we add for all particles, the terms in the bracket lead to three sums like

$$\sum_i m_i v_{\bar{x}} v'_{xi} = v_{\bar{x}} \sum_i m_i v'_{xi} = v_{\bar{x}} \sum_i m_i \frac{dx'_i}{dt} = v_{\bar{x}} \frac{d}{dt} \sum_i m_i x'_i.$$

But $\sum_i m_i x'_i$ is nothing else than M times the x of the center of

mass referred to the same axes as x'_i, y'_i, z'_i, that is, to axes through the center of mass. It is therefore 0, and we have

$$k = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\sum_i m_i v_i'^2;$$

and extending to rigid bodies, we may say that *the kinetic energy of a rigid body is the kinetic energy computed as if the body were concentrated at its center of mass, plus the kinetic energy of the motion relative to the center of mass computed as if the point were fixed.*

We have seen (p. 88) that the instantaneous motion of a body may be resolved into the instantaneous translation of one of its points, for which we here take the center of mass, and a corresponding instantaneous rotation. If ω is the instantaneous angular velocity and R the radius of gyration about the instantaneous axis, we have for the kinetic energy,

$$k = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2} mR^2\omega^2,$$

where, in general, $\bar{v}$, R, ω, and the axis of rotation are continually varying.

XV. PROBLEMS ON WORK AND ENERGY IN THE CASE OF RIGID BODIES

The following problems are intended mainly as applications of the principle of the conservation of energy.

1. Study the motion of a homogeneous sphere of mass M and radius a rolling down an inclined plane of height h and inclination i. Show that at all points the speed of its center is less than if it slid down, in the ratio of $\sqrt{5} : \sqrt{7}$. (Note that for a sphere of radius r rolling so that its center describes a straight line, $\bar{v} = r\omega$.)

2. Show that the time of rolling down the incline is $\sqrt{14\,h/5\,g} \cdot \operatorname{cosec} i$.

3. Show that for a homogeneous sphere the times of rolling down the chords of a vertical circle joining the highest point of the circle to other points of the circumference are all the same.

4. A man has a hollow iron ball and a solid aluminum ball of the same radius and weight. They are painted so that they appear the same. Explain how he can distinguish them by means of an inclined plane, telling which is which.

5. A pendulum is formed by a homogeneous ball rolling on a circular track, the plane of the circle being vertical. Show that the motion is that of the ordinary simple pendulum whose length is 7/5 times the radius of the path of the center of the ball.

6. Show that if a homogeneous right circular cylinder roll down any incline, the ratio of the kinetic energy of rotation to the kinetic energy of translation is constant. What is the constant ratio? Show that a similar statement holds for a homogeneous sphere rolling on any track.

7. In problem 11 of § 26 a particle sliding down a sphere was considered. Change this by asking at what point a rolling sphere of radius r leaves the surface of the given sphere of radius a.

8. The moon rotates about the earth, keeping the same side always toward the earth. Suppose that it always faced the same direction in space, would it have more or less kinetic energy?

32. Compound pendulum; experimental determination of moments of inertia. In the simple pendulum we have, theoretically, a *particle* moving in a circular path. We now consider *any* heavy body free to turn under the influence of gravity about a horizontal axis. Such a body is called a *compound pendulum.* If θ is the angle between the vertical plane and the plane containing the center of gravity, both planes passing through the axes, we have for the kinetic energy,

$$k = \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2} I\omega^2 = \frac{1}{2} MR^2\omega^2,$$

while the potential energy, or work done, is Mgh, h being the height through which the center of mass is raised. Let d be the distance of the center of mass from the axis. Then $h = d - d\cos\theta$, and as the sum of the energies is constant,

$$\tfrac{1}{2} MR^2\omega^2 + Mgd(1 - \cos\theta) = c.$$

Let us suppose the pendulum falls from an angle θ_0. At this point $\omega = 0$, so we have

$$0 + Mgd(1 - \cos\theta_0) = c.$$

Hence, eliminating c,

$$\tfrac{1}{2} MR^2\omega^2 = Mgd(\cos\theta - \cos\theta_0),$$

or

$$\omega^2 = \left(\frac{d\theta}{dt}\right)^2 = \frac{2\,gd}{R^2}(\cos\theta - \cos\theta_0).$$

Comparing this with the formula for the simple pendulum, (c) of § 21, we see that the motions are the same, provided $2\,gd/R^2 = 2\,g/l$, or provided $l = R^2/d$; that is, the compound pendulum

behaves just like a simple pendulum of length R^2/d. For small oscillations the period is

$$T = 2\pi\sqrt{\frac{l}{g}} = \frac{2\pi R}{\sqrt{gd}}.$$

As the position of the center of mass of a body can usually be simply determined, particularly if the body have symmetry, and hence d may be found, and as T can be readily observed, the above formula determines the radius of gyration R, and hence also the moment of inertia about the axis of suspension.

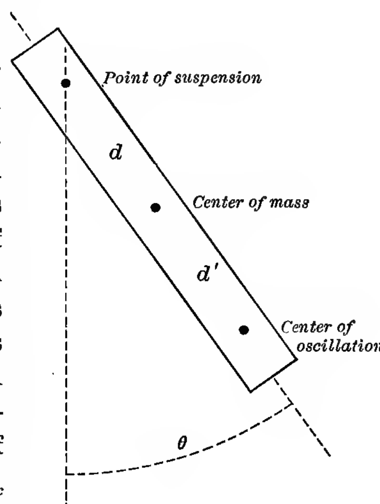


FIG. 34

The point in a line perpendicular to the axis of suspension and through the center of mass, and distant l from the axis, is called the *center of oscillation*, and is the point at which the weight of the equivalent simple pendulum would be concentrated. Let its distance from the center of mass be d'. Let us take an axis through it parallel with the axis of suspension, and call the radius of gyration about this axis R'. If R_c be the radius of gyration of the body about a parallel axis through the center of mass, we have seen (§ 30, (2), remembering that $I = MR^2$) $R^2 = R_c^2 + d^2$ and $R'^2 = R_c^2 + d'^2$, so that, subtracting, $R^2 - R'^2 = d^2 - d'^2$. Suppose now the body be suspended by the parallel axis through the center of oscillation, and let l' be the length of the equivalent simple pendulum. As $l = R^2/d$, $R^2 = ld$, and, similarly, $R'^2 = l'd'$, the above equation gives

$$ld - l'd' = (d + d')(d - d').$$

But $d + d' = l$. Hence $ld - l'd' = l(d - d')$, or $l'd' = ld'$, and since $d' \neq 0$, $l = l'$, that is, the body suspended from either axis is

equivalent to the same simple pendulum. In other words, *point of suspension and center of oscillation may be interchanged,* it being understood that the axes used are parallel.

33. General equations of motion of a rigid body. Any set of magnitudes which determine the position of a body are called the coördinates of the body. The motion of the body is determined when these coördinates are determined as functions of the time. If the motion is given through the forces that act, we shall have differential equations from which to determine these functions, in general one differential equation for each coördinate. Let us ask how many coördinates (and hence how many differential equations) are necessary to fix the position (or the motion) of a *rigid* body. To fix one point demands three coördinates. The body is then free to turn about the point. To fix the direction of a line through the point demands two more magnitudes, for instance, two of the three direction cosines (the third may be found from the fact that the sum of their squares is unity). The body is then free to turn about an axis, and if the angle through which it may be supposed to have turned is fixed, the body is fixed also. In all, then, there are six coördinates, which, however, might have been selected in a multitude of other ways. We must therefore establish six differential equations, and we shall briefly indicate how this may be done.

If we have a set of particles m_i, the coördinates of each satisfy the equations

$$\left.\begin{aligned} m_i\frac{d^2x}{dt^2}&=f_{xi}\\ m_i\frac{d^2y}{dt^2}&=f_{yi}\\ m_i\frac{d^2z}{dt^2}&=f_{zi}\end{aligned}\right\}, \tag{1}$$

where the quantities f_{xi}, f_{yi}, f_{zi} embody all forces, not only external ones but the reactions of the particles on each other. The reactions we endeavor to eliminate. Our first step is to write down the first equation (1) for each body and add them all; similarly, for the other two equations. In a rigid body, thought of as a *set* of

particles, the reactions between two particles are equal and opposite in sign, and hence drop out in the sum. So if X, Y, Z, represent the projections on the axes of the resultant of all the external forces, we have

$$\sum m_i \frac{d^2x_i}{dt^2} = X, \quad \sum m_i \frac{d^2y_i}{dt^2} = Y, \quad \sum m_i \frac{d^2z_i}{dt^2} = Z,$$

or, as
$$\sum m_i x_i = M\bar{x},$$

where M is the total mass, and $\bar{x}$ the x of the center of mass,

$$\left.\begin{aligned} M\frac{d^2\bar{x}}{dt^2} &= X \\ M\frac{d^2\bar{y}}{dt^2} &= Y \\ M\frac{d^2\bar{z}}{dt^2} &= Z \end{aligned}\right\}; \tag{2}$$

that is, the *center of mass moves as if the whole mass were concentrated there and all the external forces acted there.*

This gives us three of our differential equations of motion. Another way to eliminate the effects of internal forces is suggested by the definition of the *moment* of a force about an axis as the product of its magnitude by the perpendicular upon its line from the axis (see § 9). As the internal forces pair off into equal and opposite forces acting along the same lines, their total moments about any axis must vanish. Hence we form the moments, first of the forces acting upon m_i and about the z-axis. Calling it N_i, we have

$$N_i = m_i\left(x_i \frac{d^2y_i}{dt^2} - y_i \frac{d^2x_i}{dt^2}\right) = \frac{d}{dt} m_i \left(x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt}\right).$$

The expression

$$\begin{aligned} x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt} = r_i\left(\frac{x_i}{r_i} v_{yi} - \frac{y_i}{r_i} v_{xi}\right) &= r_i\left[v_{xi}\cos(x_i, s_i) + v_{yi}\cos(y_i, s_i)\right] \\ &= r_i v_{si}, \end{aligned}$$

where s_i denotes the direction perpendicular to the radius r_i. This is called the *moment of the velocity* about the z-axis, and

the mass times the moment of the velocity is called the *moment of momentum.* We may therefore interpret the equation

$$\frac{d}{dt} m_i \left(x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt} \right) = N_i$$

as follows: *the time rate of change of the moment of momentum of a particle about an axis is equal to the moment about the axis of the forces acting upon it.*

Compare this equation and its interpretation with the equation $m_i(d^2x_i/dt^2) = f_{xi}$, which may be written $d(m_i v_{xi})/dt = f_{xi}$. The quantity $m_i v_i$ is called the momentum of the particle m_i; the equation states that the time rate of change of the momentum in any direction is equal to the total force acting in that direction.

If we add the equations like the above for all masses, the internal forces disappear as above explained, and writing the analogous equations for the other axes, we have

$$\left.\begin{aligned} \sum \frac{d}{dt} m_i \left(y_i \frac{dz_i}{dt} - z_i \frac{dy_i}{dt} \right) &= L \\ \sum \frac{d}{dt} m_i \left(z_i \frac{dz_i}{dt} - x_i \frac{dz_i}{dt} \right) &= M \\ \sum \frac{d}{dt} m_i \left(x_i \frac{dy_i}{dt} - y_i \frac{dx_i}{dt} \right) &= N \end{aligned}\right\}, \tag{3}$$

where L, M, and N are the sums of the moments about the axes of all the external forces acting upon the body. These, with equations (2), form the required six differential equations. The last three (3) have the disadvantage, however, that they do not contain a limited number of coördinates which simply fix the position of the body in case it is rigid. This cannot be satisfactorily done without introducing moving coördinate systems, which would lead us too far at present, although it would furnish a basis for considering most interesting motions like those of rotating planets, or of gyroscopes and "tops," the latter being the name applied in mathematical physics to any rigid body with one point fixed. It is the problem which naturally follows the compound pendulum, which has two points, or an axis, fixed.

We close by pointing out the results which obtain where no external forces are present. Equations (2) and (3) admit of an integration at once, the first set giving the result valid for a rigid body or for a system of particles: *the center of mass moves in a straight line with constant velocity;* and the second, *the sum of the moments of momentum about any fixed axis is constant.*

These results find interesting application in our solar system. The student who is interested in a further study of the motion of rigid bodies is referred to more extended works on dynamics.*

ANALYSIS OF CHAPTER V

1. Definitions of displacement, translation, and rotation.
2. The instantaneous motion of a rigid body consists in an instantaneous translation and an instantaneous rotation.
3. Assumptions concerning internal forces.
4. The work and kinetic energy for a motion of translation.
5. The work and kinetic energy for a motion of rotation.
6. Definition of moments and products of inertia and the radius of gyration.
7. Two theorems enabling us to get the moment of inertia about any axis from the coefficients of inertia.
8. The work and kinetic energy for the general motion of a rigid body may be considered the sum of the work and kinetic energy, respectively, due to a translation plus the work and kinetic energy due to a rotation.
9. Compound pendulum. Experimental determination of moments of inertia. Interchangeability of point of suspension and center of oscillation.
10. General equations of motion of a rigid body. The student should gain an idea of the nature of the problem as one of reducing the number of variables from an unlimited number to six.

* For instance, Jeans, *Theoretical Mechanics;* Webster, *Dynamics.*

SUGGESTIONS AND ANSWERS

NOTE. The numerical results following are, for the most part, computed with a 30-cm. slide rule. The student should therefore not expect agreement beyond $\frac{1}{2}\%$.

Page 4, Ex. 1. Use Fig. 1.

Page 5, Ex. 3. Draw a diagram, indicating the sum vector. Next show that the order of any consecutive two of the given vectors may be changed without affecting the result. The theorem will then follow if it can be shown that any order may be made any other order by such interchanges of consecutive vectors.

Pages 9–11. II. Problems on Vectors. 3. $\sqrt{2-\sqrt{2}}\,f = 1.85f$. **4.** $v = \sqrt{(7345 - 3408\sqrt{2})} = 50.3$ ft. per sec.; 47.5° E. of N. **8 (a).** (16, 6, 6); **(d)** (2, 9.83, $-$.869). **9 (a).** 2.38, (.728, $-$.485, $-$.485); **(b)** 0, (cosines indeterminate). **10.** 22.9 ft. per sec.; 11.3° with track. **11.** Against current at angle $\arccos(a/b)$ with bank. If $a \geqq b$, impossible. Shortest time, 90° with bank; path then makes an angle $\arctan(b/a)$ with bank. **12.** 53° from the vertical. **13.** 16.7°.

Page 13, Ex. May be reduced in part to Ex. 3, p. 5.

Pages 13–16. III. Problems on Equilibrium of Concurrent Forces. 2. Consider force polygon. **4.** Use conditions (1) and Ex. 1, p. 4. 37.7, ($-$.647, $-$.727, $-$.262). **7.** Let $\bar{x}, \bar{y}, \bar{z}$ be the coördinates of O, and $x_1, y_1, z_1, x_2, y_2, z_2, \cdots, x_n, y_n, z_n$ the coördinates of the other points. Express in terms of these the projections of the vectors OA, $OB, \cdots, ON$. **8.** The components directly up the plane of the forces must be in equilibrium. **9.** As a check note that the sum of the angles should be 360°. **10.** 52 lb. **11.** Lamé's theorem may be used (problem 6); 84 and 112 lb. **12.** The chains are to be considered as of equal length. 70.7 lb. **13.** Solve a trigonometric equation. Get all roots. **15.** .5 lb. **17.** The graph is a broken straight line. **18.** The graph is partly curved and partly straight. **19.** $\mu = .3$. The reaction of the plane is equal and opposite to the other forces acting upon the mass. 10.4 lb. 16.7° with vertical. **21.** $i = \arctan 3\mu$. **22.** Height above bottom of bowl: $r(1-\cos e) = r(1 - 1/\sqrt{1+\mu^2})$.

Page 18, Ex. 4. Interpret the vanishing of the product $f_n \cdot p = 0$, supposing $f \neq 0$.

Page 19, Ex. 1. This may be done by finding the moment with respect to any point (x, y) and showing that x and y do not enter the result. Take a simple position of the axes with respect to the forces. **Ex. 2.** Apply the preceding exercise and Ex. 2, p. 18.

Page 22, Ex. 3. First show that if a set of forces is in equilibrium, so also are their projections upon any plane; do this by considering the resultant and also the moment about any perpendicular to the plane. Then show that three forces in a plane which are in equilibrium must be concurrent or parallel; do this by considering moments about the intersection of any two of the forces if any two intersect. Considering then the projections of the three given forces upon the three coördinate planes, show that the forces must be concurrent or parallel, thus proving (b). To prove (a) apply the condition for equilibrium of concurrent forces (p. 13), or, if they are parallel, consider moments about a line meeting and perpendicular to two of them. Considering then the resultant of parallel forces through the origin, (c) may be proved.

Pages 23–26. IV. Problems on Moments and the Equilibrium of Forces. 2. The resultant of the parallel forces is 15 lb. Considering moments about the lighter end of the bar, the resultant is found to be applied at a distance $3\frac{2}{3}$ ft. from this end. **3.** $2\frac{1}{12}$ ft. from the 5-lb. weight. **4.** Call the lengths of the arms d and d' and the weight of the body w. Writing the two equations expressing equality of moments and solving for w, one finds 8.48 oz. **5 (a).** 6 T.; **(b)** $5\frac{2}{3}$ and $6\frac{1}{3}$ T. **6.** $T = 3$ lb., $R = 2\frac{1}{4}$ lb., $i = 0$. **7.** 45°. **8.** The forces acting upon the gate are its weight and the reactions of the hinges, which may have various directions. **9.** $f = w(l - r \tan i) \cot i / 2l$; the center is directly above the point of contact with the floor. **10.** If the triangle of the rod and two strings be thought of as rigid, there will be seen to be two forces acting upon it, the upward pull of the hook and the downward pull of gravity upon the rod considered as acting at the mid-point. Hence the projections of a and b on a horizontal line may be shown to be equal. Then project upon a horizontal line the tensions of the strings and the weight of the bar. **11.** $T = 2cw/l$; $P = w\sqrt{1 - 2c/l + (2c/l)^2}$.

Page 28. V. Problems on the Center of Mass. 1. 10 in. from the heavy end.

Page 29, Ex. Consider whether the formulas of p. 27 hold for oblique axes; then consider the relation of the product $\Delta x \Delta y \Delta z$ to the volume of the parallelopiped with sides parallel with the axes and of lengths Δx, Δy, and Δz.

Pages 30, 36–40. VI. Problems on Centers of Mass of Continuous Bodies. 2. On the line joining a vertex to the medial point of the opposite face triangle and three fourths the distance from the vertex. **3.** $3\,r/8$ from center. **4.** Check by noting that when $a = b = c = r$ the ellipsoid becomes the sphere of the preceding problem. **5.** $2\,h/3$. **7 (a).** $2\,l/3$; **(e)** check by considerations of symmetry; **(f)** $.636\,l$. **8.** $\bar{x} = .425\,a$, $\bar{y} = .425\,b$. **10 (a).** One third the way along the diagonal (the plates should, of course, be thought of as concentrated at their centers of mass); **(b)** one third the length of the side from the middle plate. **11.** Consider either as a conical surface or as a solid cone. The result may be obtained from the answer to problem 7 (c). **13.** At center of axis. **14.** See problem 7 (d). **16.** $\bar{y} = .393$. **19.** $r/2$ from center. **20.** $2(a^2 + ab + b^2)/3(a+b)$. **22.** $\bar{x} = -4\,a/5$. **23.** $\bar{x} = (a^2/mh)\sin(h/am)$, $\bar{y} = (a^2/mh)[1 - \cos(h/am)]$, $\bar{z} = h/2$. **24.** $\bar{x} = C/M$, $\bar{y} = S/M$, where $C = 2\,a(e^{4a\pi} - 1)/(1 + 4\,a^2)$, $S = -(e^{4a\pi} - 1)/(1 + 4\,a^2)$, $M = (e^{2a\pi} - 1)/a$. **25.** $3\,h/8$ from one end. **26.** $3(b^3 + b^2 a + ba^2 + a^3)/8(b^2 + ba + a^2)$. **28.** $\bar{x} = 2\,r\sin^3 a/3(a - \sin a \cos a)$. Check by noting that as $a \doteq 0$, $\bar{x} \doteq r$. **30.** $\bar{x} = (3\,a^2 + 16\,ab + 3\,b^2)/5(a+b)$. **31.** $.652\,a$. **32.** $\bar{x} = -ar^2/2(ab - r^2)$. **34.** $\bar{y} = 4\,a/3$. **36.** $\bar{x} = -5\,a/6$. **37.** $\bar{x} = .555\,a$. **38.** $\bar{x} = .388\,a$. **40.** $\bar{x} = \bar{y} = 2\,a/5$. **41.** r/π from axis, $h/4$ from base. **42.** Fig. 14. Distance from left-hand edge $= (2\,a^2 + bc - 2\,c^2)/2(2\,a + b - 2\,c)$; Fig. 15. Distance from bottom $= (ac^2 + 2\,abc + b^2 d)/2(ac + bd)$; Fig. 16. $\bar{y} = 5.41$. **43.** $\bar{x} = .538\,a$, $\bar{y} = 1.199\,a$. **44.** $\bar{x} = 8\,a/15$, $\bar{y} = 152\,a/525$, $\bar{z} = 11\,a/120$. **45.** $\bar{x} = b^3 C/3M$, $\bar{y} = b^3 S/3\,M$ where $C = [e^{3am}(\sin a + 3\,m\cos a) - 3\,m]/(1 + 9\,m^2)$, $S = [e^{3am}(3\,m\sin a - \cos a) + 1]/(1 + 9\,m^2)$, $M = (b^2/4\,m)(e^{2ma} - 1)$. **46.** $.418\,a$. **47.** Distance from edge $= 3\,\pi r \sin a/16\,a$. Check by putting $a = \pi/2$, and also $a = \pi$; cf. problem 14. **50.** $A = 4\,\pi^2 ab$, $V = 2\,\pi^2 a^2 b$. **51.** $\bar{x} = 3\,r(1 + \cos a)/8$. **52.** In this problem the mass in question is understood to be *outside* the conical surface as opposed to problem 51. $\bar{x} = -3\,r(1 - \cos a)/8$.

53. $\bar{x} = 3\,r\sin^4 a/4(2-2\cos a-\sin^2 a\cos a)$. **54.** $\bar{x}=3\sqrt{r^2-a^2}/8$. **55.** Use $M\bar{y}=\int_0^{2a} yx\,dx$. Apply to half the plate to the left of $x=\pi a$, which half will have the same $\bar{y}$ as the whole plate. $\bar{x}=\pi a$, $\bar{y}=5\,a/6$. **57.** Use the integration tables. $\bar{x} = a\,[6\sqrt{2}-\log_e(3+2\sqrt{2})]/8\,[\sqrt{2}+\log_e(1+\sqrt{2})] = .365\,a$. **58.** By "cone" is meant conical surface, and all matter is removed from the sphere which lies within the indefinitely prolonged conical surface. $\bar{x} = r\sin^2 a$.

Pages 46–48. VII. Problems on Rectilinear Motion. 2 (a). $t=-1$, $s=6$; (b) points to left of $s=6$; (c) forward to $s=6$, then back; (d) part of line to left of $s=6$, twice; (e) to the left; (f) for large negative t, the point is far to the left with large positive but diminishing velocity; for large positive t, the point is far to the left with large negative and numerically increasing velocity. **20, 21.** The equations give more than one value of s for a given value of t. What would this mean? **22.** $s=(kt^2+2)/2$. **24.** $s=2(e^{kt}+k-2)/k$.

Pages 52–54. VIII. Problems on Bodies moving in Straight Lines under the Action of Given Forces. 2. 403.6 ft. **5.** Determine the height h of the body at its highest point and thus answer the problem. **6.** 53 ft. **7.** The weight of only one of the bodies is effective in producing motion, while the inertia of both bodies resists the motion. They are, of course, supposed to start from rest. (c) $v=242$ cm. per sec. **8.** Write an equation for each body in which its acceleration is equated to its mass times the resultant of the forces acting upon it, namely gravity and the tension of the cord. From these equations determine the unknown acceleration. **10.** About 3 ft. **13.** The path of the particle must bisect the angle between the vertical and the perpendicular to the line on to which it falls. **14.** About 150 lb. **15.** Six times as high. 5.18 sec. **16.** About 10.3°. **17.** 6012 ft. **18.** 410 ft. **19.** 5.7 T., or about 57 lb. per T. **20.** About 1268 sec., or 21.1 min. 25,950 ft. per sec. **22.** 1 ft. **23.** About 7.8 kgm. **24.** 25,950 ft. per sec. 28.1 min. **26.** $k=17.6$; 10.15 ft. per sec. **27.** Velocity of bullet is about 3.4 times that of raindrop. **30.** Velocity upon leaving table is $\sqrt{5/g}\,\log_e(5+\sqrt{24})=.904$ ft. per sec.

Pages 73–76. X. Problems on Bodies moving in a Plane under the Action of Forces. 2. 106 ft. per sec. **4.** 6.2°. **6.** 2.63 ft. 10.3° from vertical. **7.** Solve the problem from the standpoint that the stream of water is sent from a fixed point, and consider the area above the

level of that point only. 1100 sq. m. **9.** Use equation of envelope. **16.** By $\frac{1}{240}$ of its length. **19.** The centrifugal force is mv^2/ρ *poundals*, if m is measured in pounds. 1.34 lb. **20.** 433 T. **22.** 14.9°. **23.** About 1/291 and 17. **24.** The data lead to a result 45 ft. As this is a very short throw, how should the data be changed? **25.** $4\,a^2mv^2/(y^2+4\,a^2)^{\frac{3}{2}}$ poundals. **28.** $y = y_0 + (kh - g)/k\,(1 - \cos\sqrt{k}t)$.

Pages 78, 79. XI. Problems on Work and Power. **3.** 37.9 h. p. **5.** .14 ft. lb. **6.** $5\frac{1}{4}$ h. p. **9.** 800 h. p. **10.** $(b + a - 2\,l)(b - a)\lambda/2\,l$ ft. lb. **11.** 1.21 h. p.

Page 84. XIII. Problems on Energy. (NOTE. In these problems the student must be careful about units. In the derivation of the formula for $k = \frac{1}{2}\,mv^2$, p. 83, the equations $f_x = m\,(dv_x/dt)$, $f_y = m(dv_y/dt)$ hold if m is measured in pounds and f_x and f_y in *poundals;* and also if m is measured in engineering units (weight $\div\ g$) and f_x and f_y in pounds. Thus if m is measured in pounds, $k = \frac{1}{2}\,mv^2$ *foot poundals;* if m is measured in engineering units, $k = \frac{1}{2}\,mv^2$ *foot pounds.*) **2.** 19,100 ft. lb. **3.** 368,000 ft. lb. **4.** To a point 24 ft. higher than the foot of the hill. **6.** 4%. **7.** $v_0^2/2\,g$ ft. above the lowest point if the rod is long enough. **10.** 5850 h. p. **11.** At a height $2\,a/3$ above the level of the center.

Pages 95–98. XIV. Problems on Moments and Products of Inertia and Radii of Gyration. **5.** $l^3/12$; $l^3/24$; $l^3/3$. **6.** $A = B = l^4/12$, $C = l^4/6$, $D = E = F = 0$; $I_d = l^4/12$; $I_c = 2\,l^4/3$; $I_m = 5\,l^4/12$. **8.** $A = B = \pi r^4/4$, $C = \pi r^4/2$; $r = l/\sqrt[4]{3\,\pi} = .57\,l$. **9.** $r = .63\,l$. **11.** $C = 2\,ar^3$. **12.** $A = \pi ab^3/4$, $C = \pi ab\,(a^2 + b^2)/4$; $I_{fm} = \pi ab\,(4\,a^2 - 3\,b^2)/4$; $I_{fp} = \pi ab\,(5\,a^2 - 3\,b^2)/4$. **13.** If $A = f(a, b, c)$, then $B = f(b, c, a)$, $C = f(c, a, b)$. $A = 4\,\pi abc\,(b^2 + c^2)/15$. **15.** $C = 35\,\pi a^4/16$, $A = 21\,\pi a^4/32$. **16.** $A = abc\,(b^2 + c^2)/12$; $I_a = abc\,(b^2 + c^2)/3$; $I_d = abc\,(a^2b^2 + b^2c^2 + c^2a^2)/6\,(a^2 + b^2 + c^2)$. **17.** $abc[2\,a^2(b^2 + c^2) + b^2c^2]/6(b^2 + c^2)$. **19.** $C = \pi ar^4/10$. The result of problem 8 may be used to advantage here. **20.** $A = B = \pi ar^2(a^2 + 3\,r^2)/12$, $C = \pi r^4a/2$. **21.** $A = 43\frac{1}{3}$, $B = 79\frac{2}{3}$, $C = A + B = 123$, $F = -15$. **25.** $r' = .235\,r$.

Pages 101–102. XV. Problems on Work and Energy in the Case of Rigid Bodies. **6.** $K_r : K_t = 2:1$ for cylinder and $5:2$ for sphere. **7.** Difference in heights of centers of spheres is $10(a + r)/17$ at time of leaving.

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[The numbers used below refer to pages.]

ANNOUNCEMENTS